A PIANO MODEL INCLUDING LONGITUDINAL STRING VIBRATIONS

Balázs Bank and László Sujbert

Department of Measurement and Information Systems
Budapest University of Technology and Economics
{bank|sujbert}@mit.bme.hu

ABSTRACT

In this paper a mixed-paradigm piano model is presented. The major development is the ability of modeling longitudinal string vibrations. Longitudinal string motion is the reason for the metallic sound of low piano notes, therefore its modeling greatly improves the perceptual quality of synthesized piano sound. In this novel approach the transversal displacement of the string is computed by a finite-difference string model and the longitudinal motion is calculated by a set of second-order resonators, which are nonlinearly excited by the transversal vibration. The soundboard is modeled by a multi-rate filter based on measurements of real pianos. The piano model is able to produce high-quality piano sounds in real-time with about 5–10 note polyphony on an average personal computer.

1. INTRODUCTION

As the piano has hundreds of strings, the bottleneck of piano modeling lies in the strings. For string modeling the digital waveguide [1] is by far the most efficient approach. Instead of discretizing the wave equation, it discretizes its traveling-wave solution. Since in most of the cases the string motion has to be computed correctly at the excitation and observation points only, the model is reduced to a delay line and a low-order ($N = 10..20$) filter in a feedback loop.

Already high-quality real-time physical models of the piano have been presented based on digital waveguide modeling [2, 3, 4, 5]. However, in these models the effect of longitudinal vibrations was neglected. The quality of these models can be greatly improved by including the longitudinal motion of piano strings. In the low range of real pianos the pitch of the longitudinal components can be perceived by the listener, and the subjective quality of the instrument is highly dependent on the frequency of these modes [6], pointing out that the longitudinal string motion has an important perceptual effect.

The longitudinal vibration of piano strings is made up of the free vibration of longitudinal modes and the forced motion excited by the transversal displacement. The spectral peaks corresponding to the forced motion are called “phantom partials” [6]. We have presented a detailed analysis on how these partials are generated in [7]. Fig. 1 shows an extract of the spectrum of a G_1 piano note, recorded at 2 m distance from the piano. The phantom partials are clearly visible between the transversal partial series, so is one longitudinal mode (marked by circle), which has even larger amplitude than the neighboring transversal ones.

Accordingly, it would be highly beneficial to incorporate the longitudinal modes in the efficient digital-waveguide based piano models. Borin [8] have amended his real-time piano model with

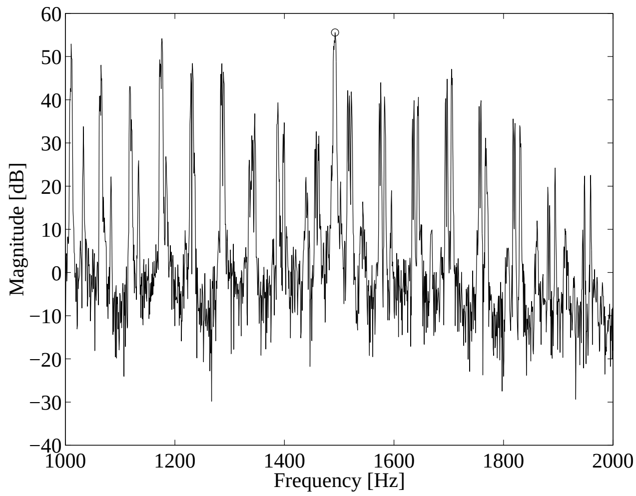

Figure 1: *Spectrum of the first second of a G_1 piano note (forte playing) between 1 and 2 kHz. A prominent longitudinal mode is marked by a circle.*

independent digital waveguides for the longitudinal polarization. We have also made experiments with similar solutions [7]. In these models the longitudinal modes are excited during the hammer-string contact only, therefore the forced longitudinal motion is not simulated. These simple models capture some aspects of low piano tones, but sound unnatural. The longitudinal modes sound separated from the transversal ones, unlike in real piano sounds and in finite-difference simulations. We suppose that the key to having coherence between transversal and longitudinal components is that the longitudinal vibration is continuously excited by the transversal one.

In the next Sections, we will present a new approach to string modeling, which is able to model the continuous interaction between the transversal and longitudinal polarizations efficiently. After the derivation of the basic equations a finite-difference model is described. This is followed by the new approach, which is basically the simplification of the finite-difference model, having the same perceptual quality at around 10% computational cost. Then, parameter estimation techniques are described, and simulations presented. Possible directions of future research conclude the paper.

2. THE BASIC EQUATIONS

A real piano string is vibrating in two transversal planes, and in the longitudinal direction as well. Principally, piano hammers excite one transversal polarization of the string, the other two are gaining energy through coupling. These polarizations interact with each other as a result of nonlinear behavior of the string.

For simplicity, let us assume that the string is vibrating in one plane, thus, one transversal and one longitudinal polarization is present. We will see later that modeling these two polarizations produces high-quality piano sound. Naturally, the model can be easily extended to comprise two transversal polarizations.

When a transversal displacement occurs on the string, the string elongates. This results in a force exciting a longitudinal wave in the string. The longitudinal wave modulates the tension of the string, which influences the transversal vibration. Note that throughout this Section losses and dispersion are not considered, since now we are mainly interested to understand the coupling between the two polarizations. (A more precise derivations in a vector-based formulation can be found e.g., in [9]).

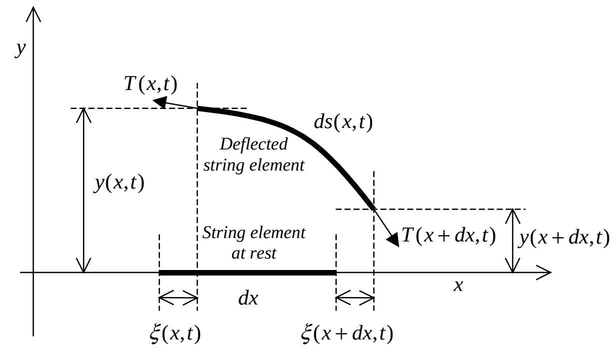

Figure 2: *The string element.*

The element of length dx at equilibrium will have the length ds, as depicted in Fig. 2, which is calculated as follows:

$$\begin{aligned} ds^2(x,t) &= (\xi(x+dx,t) - \xi(x,t) + dx)^2 + \\ &+ (y(x+dx,t) - y(x,t))^2 \end{aligned} \tag{1}$$

As dx is infinitesimally small, the differences are substituted by differentials:

$$ds = \sqrt{\left(\frac{\partial \xi}{\partial x} + 1\right)^2 dx^2 + \left(\frac{\partial y}{\partial x}\right)^2 dx^2} \tag{2}$$

where $y = y(x,t)$ and $\xi = \xi(x,t)$ are the transversal and longitudinal displacements of the string with respect to time t and space x. As the length of the element changes varies the tension $T = T(x,t)$ (which equals to T_0 at rest) of the string according to the Hooke's law:

$$T = T_0 + ES\left(\frac{ds}{dx} - 1\right) \tag{3}$$

where E is the Young's modulus and S is the cross-section area of the string. By substituting Eq. (2) into Eq. (3) the string tension can be approximated as:

$$T \approx T_0 + ES\left(\frac{\partial \xi}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right) \tag{4}$$

As the segment ds is nearly parallel to the x axis, the longitudinal force on the segment ds is the difference of the tension at the sides of the segment:

$$F_l \approx \frac{\partial T}{\partial x} dx \approx ES\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2}\frac{\partial\left(\frac{\partial y}{\partial x}\right)^2}{\partial x}\right) dx \tag{5}$$

This force acts on a mass μdx, where μ is the mass per unit length. Accordingly, the longitudinal vibration is approximately described by the following equation:

$$\mu \frac{\partial^2 \xi}{\partial t^2} = ES\frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2} ES \frac{\partial\left(\frac{\partial y}{\partial x}\right)^2}{\partial x} \tag{6}$$

which is the standard wave equation with an additional force term depending on the transversal vibration of the string according to a second-order nonlinearity. Note that the transversal string motion can only excite the longitudinal vibration if the square of the string slope is significant, i.e., the transversal displacement is relatively large.

After similar derivations, the wave equation for the transversal motion can be written as follows:

$$\mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2} + ES \frac{\partial\left(\frac{\partial y}{\partial x}\frac{\partial \xi}{\partial x}\right)}{\partial x} \tag{7}$$

which is again a standard wave equation with an additional force term depending on the product of transversal and longitudinal string slope. Consequently, the longitudinal vibration influences the transversal one at large displacements only.

By looking at Eqs. (6) and (7) we can conclude that the coupling of transversal and longitudinal string motion depends on the magnitude of vibration according to a square law and that the coupling is bi-directional.

3. FINITE-DIFFERENCE MODELING

A straightforward choice for computing the solution of the above equations is finite-difference modeling, but first the partial differential equations Eqs. (6) and (7) have to be extended by additional terms. The equation for transversal vibration including dispersion and frequency-dependent losses takes the form:

$$\begin{aligned} \mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2} - ES\kappa^2 \frac{\partial^4 y}{\partial x^4} - 2b_1 \mu \frac{\partial y}{\partial t} + 2b_2 \mu \frac{\partial^3 y}{\partial^2 x \partial t} + \\ + ES \frac{\partial\left(\frac{\partial y}{\partial x}\frac{\partial \xi}{\partial x}\right)}{\partial x} \end{aligned} \tag{8}$$

which is similar to the equation used in [5, 10], except the rightmost term corresponding to the effect of longitudinal vibration to the transversal one. The κ sign in the dispersion term refers to the radius of gyration of the string, and the constants b_1 and b_2 determine the decay time τ_k of partial k:

$$\tau_k = -\frac{1}{b_1 + b_2 \omega_k^2} \tag{9}$$

where ω_k is the angular frequency of the corresponding partial [5, 10].

Likewise, the equation for longitudinal vibration has to be completed by frequency-dependent loss terms similar to what was used for transversal vibration in [5, 10]:

$$\mu \frac{\partial^2 \xi}{\partial t^2} = ES\frac{\partial^2 \xi}{\partial x^2} + 2b_{1l}\mu \frac{\partial \xi}{\partial t} + 2b_{2l}\mu \frac{\partial^3 \xi}{\partial^2 x \partial t} + \frac{1}{2} ES \frac{\partial\left(\frac{\partial y}{\partial x}\right)^2}{\partial x} \tag{10}$$

where b_{1l} and b_{2l} set the decay times of the longitudinal modes in the same way as b_1 and b_2 for the transversal vibration (see Eq. (9)). The longitudinal modal frequencies are not in a perfect harmonic series in real pianos [7]. The simplest (although not physically meaningful) way of achieving this effect in the model is having a not uniform mass density $\mu(x)$ in Eq. (10) along the dimension x.

As the string is assumed to be hinged at both ends, the corresponding boundary conditions become [11]:

$$y(0,t) = y(L,t) = \xi(0,t) = \xi(L,t) = 0$$
$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=L} = 0 \tag{11}$$

The solution of the partial differential equations is computed on a grid $x_m = m\Delta x$, $t_n = n\Delta t$ by substituting the differentials by finite differences:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_m,t_n} \approx \frac{u(x_{m-1},t_n) - 2u(x_m,t_n) + u(x_{m+1},t_n)}{\Delta x^2}$$
$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{x_m,t_n} \approx \frac{u(x_m,t_{n-1}) - 2u(x_m,t_n) + u(x_m,t_{n+1})}{\Delta t^2}$$
$$\left.\frac{\partial u}{\partial t}\right|_{x_m,t_n} \approx \frac{u(x_m,t_n) - u(x_m,t_{n-1})}{\Delta t} \tag{12}$$

where u is used as a general notation for either ξ or y. The fourth-order term in Eq. (8) is computed by using the first line of Eq. (12) twice. The string model is then connected to a simple finite-difference hammer model simulating the hammer-string interaction [11]. The resulting finite-difference equations are explicit, meaning that the next values of $y(x_m,t_n)$ and $\xi(x_m,t_n)$ can be easily computed from the previous values without the need of iterations.

We have developed such a model consisting of 100 string elements [7]. The finite-difference string model produces high sound quality, but requires a large amount of computation. It is mainly because of the high traveling speed c_l in the longitudinal direction, which makes large sampling rates (e.g, f_s = 500 kHz) necessary in order to avoid numerical instability. With today's personal computers this would mean one note polyphony (i.e., monophony). However, for experimental purposes, this kind of approach is still beneficial. For example, a commercial computer program based on similar principles was written by Bernhard [12], to help piano builders in scale design.

A complete finite-difference string model made it possible to experiment whether it is reasonable to neglect the coupling from longitudinal to transversal motion or not. We have found that although the produced waveforms are slightly different, the perceptual difference is insignificant. In general, this means that it is enough to model the coupling from transversal to longitudinal vibrations, allowing large simplifications, which will be presented in Sec. 4.

However, if the transversal-to-longitudinal excitation force (the rightmost term of Eq. (6)) has strong peaks at the longitudinal modal frequencies, the longitudinal modes may reach extreme amplitude levels if the longitudinal-to-transversal coupling is neglected. This would not happen if the coupling from longitudinal motion to the transversal one was also realized, since in that case the longitudinal mode would diminish the amplitude of those transversal partials from where it originates [7]. On the other hand, piano builders try to avoid these constellations anyway, therefore we can do the same by setting the longitudinal modal frequencies different from the peaks of the transversal-to-longitudinal excitation force (see also Sec. 5).

4. THE NEW APPROACH

The starting point of our composite model is the finite-difference approach described in Sec. 3, as the transversal displacement needs to be known precisely for each point along the string for computing the transversal-to-longitudinal coupling precisely.

The basic idea allowing the simplification of the original finite-difference string model described in Sec. 3 is that the longitudinal displacement should be known at the termination only, since the feedback from the longitudinal motion to the transversal one is neglected. Therefore, there is no need for a finite-difference model for computing longitudinal vibrations, which eliminates the problem of high (e.g., 500 kHz) sampling rates.

Accordingly, the transversal string displacement $y(x,t)$ is computed by a finite-difference model similar to the Eq. (8) excluding the coupling from longitudinal vibrations at audio sampling rate ($f_s = 44.1$ kHz). On the contrary, the longitudinal motion is described by it's modal form [13]:

$$\xi(x,t) = \sum_{k=1}^{N} a_k \sin(\pi \frac{kx}{L}) \cos(2\pi f_k t) e^{-\frac{t}{\tau_k}} \tag{13}$$

where k is the mode number, N is the total number of modes to be computed, L is the length of the string, a_k, f_k, and τ_k are the amplitude, frequency, and decay time of mode k, respectively. The force at the bridge $F_{l,br}(t)$ can be approximated by:

$$F_{l,br}(t) \approx ES \left.\frac{\partial \xi(x,t)}{\partial x}\right|_{x=L} =$$
$$\frac{\pi}{L} \sum_{k=1}^{N} a_k k (-1)^{k-1} \cos(2\pi f_k t) e^{-\frac{t}{\tau_k}} \tag{14}$$

which can be implemented as second-order resonators in parallel.

Eq. (14) describes only the excitation-free motion of the longitudinal modes. Regarding the forced motion, first the excitation force distribution $F_{l,exc}(x,t)$ is computed from the transversal displacement according to Eq. (6):

$$F_{l,exc}(x,t) = \frac{1}{2} ES \frac{\partial \left(\frac{\partial y(x,t)}{\partial x}\right)^2}{\partial x} \tag{15}$$

Then, the force input of mode k is calculated by the following way:

$$F_{l,k}(t) = \int_{x=0}^{x=L} \sin(\pi \frac{kx}{L}) F_{l,exc}(x,t) dx \tag{16}$$

which is the scalar product of the force distribution $F_{l,exc}(x,t)$ and the modal shape of mode k [13].

The equations were presented in continuous time for clarity. In the synthesis model the differetial and integral operations are substituted by finite difference and summation. The computationally heavy part of longitudinal-vibration simulation lies in Eqs. (15) and (16). Especially the load of Eq. (16) is heavy, since it means that the force input $F_{l,k}(t)$ is computed for all the modes ($N \approx 10$ in practice) separately. Therefore, further simplifications are necessary.

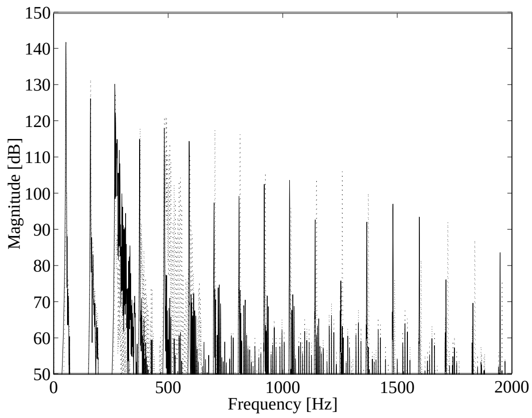

Figure 3: *The spectrum of the excitation force for longitudinal mode 5 ($F_{l,5}$ – solid line), and mode 9 ($F_{l,9}$ – dotted line) for comparison. These modes contribute to odd phantoms.*

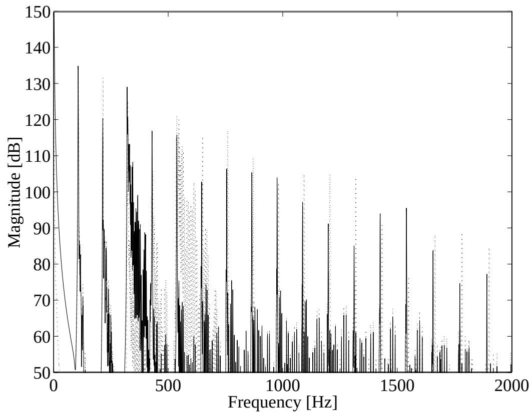

Figure 4: *The spectrum of the excitation force for longitudinal mode 6 ($F_{l,6}$ – solid line), and mode 10 ($F_{l,10}$ – dotted line) for comparison. These modes give a rise to even phantoms.*

The excitation spectrum (the Fourier transform of $F_{l,k}(t)$) of all the odd and all the even longitudinal modes are very similar, respectively. It can be seen in Figs. 3 and 4, that the only difference is that the frequency peaks are slightly shifted as a function of mode number k because of the inharmonicity of the string [7]. The amplitudes are also somewhat different, but the general envelopes are of quite similar structure. Therefore, it is a logical choice to substitute the excitation force $F_{l,k}(t)$ of all the odd longitudinal modes by the excitation force of one odd longitudinal mode (e.g., $F_{l,k}(t) = F_{l,5}(t)$ for odd k). The same can be done for the even longitudinal modes. However, it is important to incorporate at least one odd and one even modal shape, since odd longitudinal modes give a rise to odd phantom partials, and even modes to even phantoms. Having only one modal shape in the model would lead to an excitation spectrum with odd or even harmonics only. Accordingly, the model can be simplified by computing the force input for two modes (e.g., $F_{res} = F_{l,5} + F_{l,6}$, but any other odd and even mode would do) and using this as a common excitation for all the resonators. This leads to almost identical perceptual results compared to the full model of Eq. (16) for $k = 1..N$.

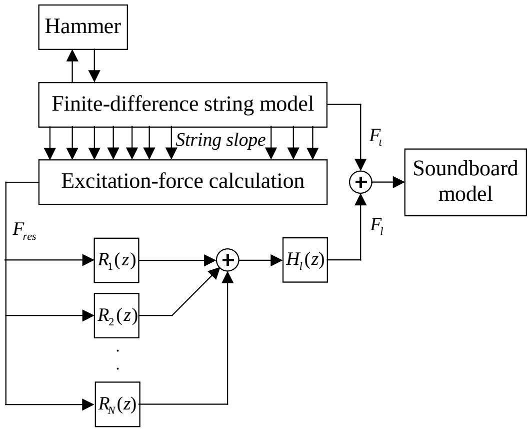

Figure 5: *The composite string model applying finite differences and resonators.*

The string model is depicted in Fig. 5. It can be seen that the transversal string displacement is computed by a finite-difference string model, which is excited by a finite-difference hammer model [14]. The transversal force F_t at the bridge is the force at the right-hand side termination.

Then the excitation force of the resonators F_{res} is computed by squaring the string slope at each point, differentiating along the dimension x (i.e., approximating Eq. (15)) and computing a scalar product with the modal shape of two consecutive longitudinal modes (similarly to Eq. (16)). The longitudinal force at the bridge is then computed by feeding the excitation signal to a resonator bank $R_1(z)..R_N(z)$. This signal is filtered by $H_l(z)$ to take into account that the soundboard has a different response to longitudinal bridge deflection compared to the transversal one. We have found that already a simple differentiation $H_l(z) = 1 - z^{-1}$ produces good results.

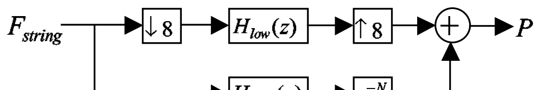

Figure 6: *The multi-rate soundboard model [4].*

The force signals of the two polarization F_t and F_l are added and sent to a soundboard model based on multi-rate filtering [4]. The soundboard model is depicted in Fig. 6. The string signal is split into two parts: the signal below 2.2 kHz is downsampled by a factor of 8 and filtered by a 4000 tap (meaning 360 ms length at $f_s/8$) FIR filter $H_{low}(z)$ precisely synthesizing the amplitude and phase response of the soundboard for the low frequencies. The signal above 2.2 kHz is filtered by a 1000 tap (ca. 20 ms at $f_s = 44.1$ kHz) FIR filter. This simplification in the high-frequency chain can be done because the higher modes of the soundboard decay faster than the lower ones, while the ear is also less sensitive in this region. The signal of the high frequency chain is delayed by N samples to compensate for the latency of decimation and interpolation filters of the low frequency chain. The sound produced by this model is indistinguishable from that calculated by a 16000 tap FIR filter directly implementing the soundboard impulse response.

5. PARAMETER ESTIMATION

The parameters of the hammer model are taken from [14]. The lengths of the strings are measured on a real piano. Instead of measuring the mass of the string, the mass density μ is computed from the tension T_0 (set to 700 N), the length, and the fundamental frequency. String inharmonicity and the loss parameters of the transversal vibration are estimated from recorded piano tones by polynomial regression. As for the losses, the method is basically the first step of the one-pole filter design method presented in [15].

The frequencies of the longitudinal modes (i.e., the frequencies of $R_1..R_N$) can be set according to the spectra of real piano tones. However, the peaks of longitudinal modes cannot be easily found between the transversal ones automatically, which results in a huge amount of work. Alternatively, the longitudinal modal frequencies can be set in a way that they should correspond to a tone which is in harmonic relationship to the transversal vibration (e.g., the longitudinal component sounds four octave higher than the transversal one). This is what piano builders wish to achieve in real pianos as well [6]. However, these frequencies should not lie on the peaks of the excitation force F_{res} (i.e., the solid lines in Figs. 3 and 3), since that would lead to undesirable ringing. This can be done automatically by computing the spectrum of the excitation signal F_{res} and shifting those longitudinal mode frequencies which are too close to some of the peaks. The decay time of the resonators were set to around 0.1 sec in most of the cases. The ratio between the transversal and longitudinal vibration is controlled by the amplitudes of the resonators and was set manually.

The parameters of the soundboard model were taken from force-hammer measurements of a real piano soundboard [4]. The filters $H_{low}(z)$ and $H_{high}(z)$ are computed as follows: first a 16000 tap target impulse response $H_t(z)$ is calculated by measuring the force–pressure transfer function of the soundboard. This is lowpass-filtered and downsampled by a factor of 8 to produce an FIR filter $H_{low}(z)$. The impulse response of the low frequency chain is now subtracted from the target response $H_t(z)$ providing a residual response containing energy above 2.2 kHz. This residual response windowed to a shorter length (1000 tap).

6. RESULTS

In Fig. 7 the spectrum of a synthesized G_1 piano tone is depicted with 5 m/s hammer impact speed (forte playing). The phantom partials and the free response of the second longitudinal mode (marked by a circle) are clearly visible between the transversal partials and sometimes reach higher amplitude levels than the transversal partials (e.g., around 1.5 kHz). It is important to note that the most important part of the longitudinal vibration is the forced motion (phantom partials) and not the free mode marked by a circle. For comparison, the spectrum coming from the transversal string vibration is displayed in Fig. 8. Note that without longitudinal modeling the spectrum is clean and contains a quasi-harmonic series only. Both signals were generated including the multi-rate soundboard in the model.

Comparing Fig. 7 to Fig. 1 shows that the spectrum of the synthesized piano tone is much closer to the original if longitudinal string motion is also considered in the modeling. There are still some differences between the synthesized and original. However, this is not considered as a drawback, since in physics-based sound synthesis the goal is rather to develop a model which has a realistic piano sound than to imitate a particular type of piano.

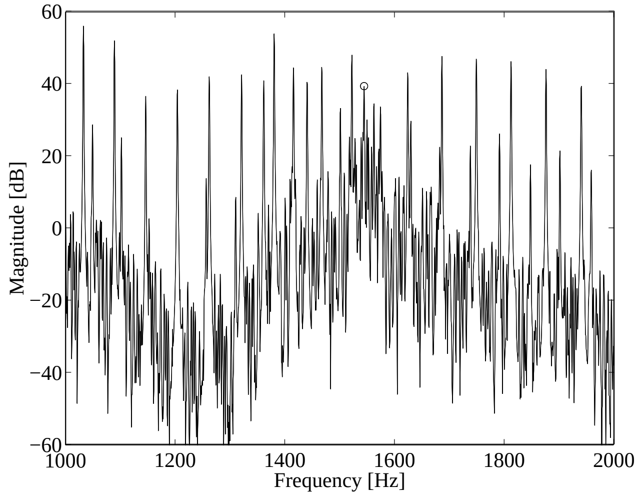

Figure 7: *Spectrum of a simulated G_1 note computed by the composite model of Fig. 5. The peak of the second longitudinal mode (modeled by $R_2(z)$ in Fig. 5) is marked by a circle.*

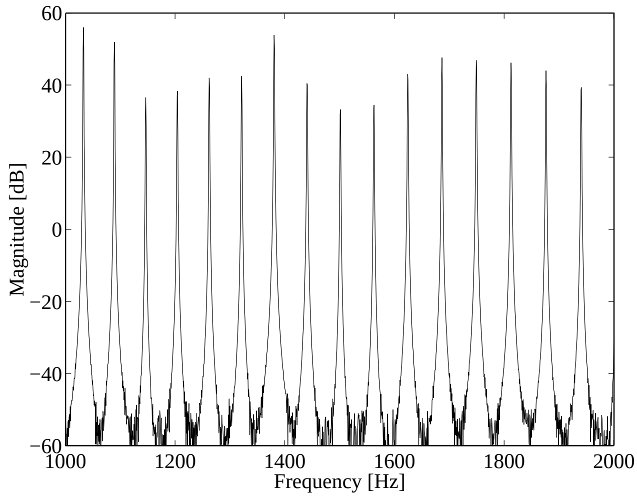

Figure 8: *Spectrum of a simulated G_1 note without longitudinal modeling (F_l was set to zero in Fig. 5).*

Although it is not depicted, the model responds to the variations of playing dynamics (piano to forte) realistically. The longitudinal components become significant at high dynamic levels only similarly to real pianos.

The model is capable of producing similar sound quality compared to the finite-difference model of Sec. 3 at around 10% of computational cost. Computational savings are achieved by eliminating the need of huge sampling rates, which were necessary to assure numerical stability. This is the reason why the perceptual quality of the finite-difference model is still preserved: the major difference is that the new model does not compute anything above 20 kHz, which we would not hear anyway. The new approach has an other advantage over the finite-difference method: the flexibility of setting the longitudinal modal frequencies.

Compared to earlier digital waveguide or finite-difference based piano models the sound quality improved significantly for low pi-

ano tones. The computational requirements are 10–20 times higher than that of a digital-waveguide based string model, still allowing 5–10 note polyphony in real-time on an average PC, in C++ implementation. Practically, this means that those piano pieces can be played which do not require sustain pedal. Sound examples can be listened at:
`http://www.mit.bme.hu/~bank/publist/dafx04`

7. CONCLUSIONS AND FUTURE WORK

In this paper a novel approach was presented for modeling the longitudinal vibration of piano strings. The method is based on a finite-difference string model for transversal vibrations, driving second-order resonators for longitudinal-vibration simulation. Large computational savings have been achieved compared to the complete finite-difference string model with no loss of sound quality, allowing the use of the method in real-time. Simplifications were done along perceptual lines, i.e., those factors were neglected, which have no significant effect on the produced sound.

As for further improvements in sound quality, the force-pressure transfer function of the soundboard for the longitudinal polarization could be measured on real pianos. This would allow the use of a separate soundboard model for the longitudinal polarization. Alternatively, a precise shaping filter $H_l(z)$ in Fig. 5 could be designed. Another natural choice can be incorporating the other transversal polarization in the model.

The main area of future investigations should be reducing the computational complexity. One attempt to this can be substituting the finite-difference model in Fig. 5 by a digital waveguide for computing the transversal vibration. Now the difficulty is that the digital waveguide in its efficient form is intended to compute the string motion at the observation point (the bridge in this case) only. Therefore the modal shapes will be different from that of a real string, since the wavetrains will close through the dispersion filter at the termination. A solution can be having multiple (50..100) observation points by distributing the all-pass filters between the delay elements, similarly to what was done in the case of the kantele [16] for a different reason, namely, for tension-modulation modeling. Unfortunately, this leads to computational requirements almost as heavy as of the finite-difference string model.

Larger computational savings could be achieved by concentrating more on the perceptual aspects of longitudinal vibration. As little is known about how these components are perceived, this calls for psychoacoustic studies.

8. ACKNOWLEDGEMENTS

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