# MODAL-TYPE SYNTHESIS TECHNIQUES FOR NONLINEAR STRINGS WITH AN ENERGY CONSERVATION PROPERTY

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**ABSTRACT**

There has recently been increased interest in the modelling of string vibration under large amplitude conditions, for sound synthesis purposes. A simple nonlinear model is given by the Kirchhoff-Carrier equation, which can be thought of as a generalization of the wave equation to the case for which the string tension is "modulated" by variations in the length of the string under deformation. Finite difference schemes are one means of approach for the simulation of nonlinear PDE systems; in this case, however, as the nonlinearity is spatially invariant, the solution may be broken down into sinusoidal components, much as in the linear case. More importantly, if time discretization is carried out in a particular way, it is possible to obtain a conserved energy in the numerical scheme, leading to a useful numerical stability guarantee, which can be difficult to obtain for strongly nonlinear systems. Numerical results are presented.

## 1. INTRODUCTION

When it is desired to simulate the transverse motion of a vibrating string, in a single polarization, a simple starting point is the 1D wave equation [1]. There are many approaches to the simulation of such an equation: the most straightforward makes use of finite difference approximations [2, 3, 4]. In this case, the defining PDE is discretized, leading to a solution approximated at various grid points, and at a finite set of time instants. Digital waveguides [5] represent a particularly elegant special case of a finite difference method, for which the numerical solution may be calculated as a sum of discrete travelling waves, at a greatly reduced computational cost. When the wave equation is complemented by additional terms which model such perceptually crucial effects as loss and dispersion [6, 7], finite difference schemes may still be used; for certain schemes, an equivalent digital waveguide formulation also follows [8], and it is also possible to construct quasi-physical waveguide structures which remain relatively cheap, computationally speaking [9].

Because the wave equation (and its extensions as mentioned above) is linear and time-invariant, a complete description is available in the frequency domain; the behavior of the string may be broken down into independent contributions from various modes, each of a particular shape and frequency. The shapes and frequencies are strongly dependent on the particular type of boundary conditions applied. Synthesis based on such a decomposition is often referred to as *modal synthesis* [10]; it may be extended to deal with general linear and time-invariant systems. If the system is, in addition, spatially-invariant (i.e., if there is no material parameter variation), all solutions may be expressed in terms of modal contributions, the shapes of which are complex exponentials; the wave equation for a string of constant density is of this form, but the vocal tract, for instance, is not (it still possesses modes whose time-dependence is complex exponential, but these are not spatially complex exponentials).

Under high-amplitude conditions, the 1D wave equation is no longer a good model of transverse wave propagation. The most general models of string vibration are nonlinear, and involve pointwise coupling among the two transverse polarizations and the longitudinal displacement [1, 11]. Under certain assumptions [12], generally valid for many types of strings which appear in a musical setting, transverse motion in a single polarization may be decoupled, and a simplified nonlinear equation, sometimes called the Kirchhoff-Carrier equation [13, 14], results. In this model, the effect of the nonlinearity is global—in other words, though nonlinear, the equation remains spatially shift-invariant. Such a model has served as the starting point for extensions of digital waveguides [15, 16, 17], as well as finite difference schemes [18]. Due to the spatial invariance of the nonlinearity, one might expect that a modal-type description will be available for this nonlinear model; although the string does not possess modes as such, a breakdown of the string into simple sinusoidal shapes suggests a numerical simulation approach which is similar to modal synthesis. One of the great benefits of such an approach is that it is possible to arrive at a useful stability condition on the numerical scheme; stability-checking machinery such as von Neumann type analysis [19, 20, 21] is generally not valid in the nonlinear case. Such analysis, as in the case of finite difference schemes, relies on strict energy conservation properties [22, 21].

In Section 2, we present the Kirchhoff-Carrier equation, as well as its expansion into a first order system, and briefly review its energy conservation properties. In Section 3, we first present a Fourier decomposition of the solution to the system, and show how this leads to a system of ordinary differential equations, which may be discretized in such a way as to yield a conserved energy-like quantity. We then discuss the conditions under which this conserved quantity leads to a numerical stability guarantee, and conclude with a look at spurious oscillatory behavior and some implementation details. Numerical simulation results are presented in Section 4.

## 2. THE KIRCHHOFF-CARRIER EQUATION

The Kirchhoff-Carrier equation, as mentioned above, is a good first approximation to nonlinear behavior of a string, in a transverse

polarization. It can be written as

$$\rho\frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{EA}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Here, $u(x,t)$ is the transverse string displacement, $t \geq 0$ is a time variable, and $x \in [0, L]$ is a space variable. The string is characterized by the parameters $\rho$ (linear mass density), $T_0$ (applied tension), $E$ (Young's modulus), and $A$ (cross-sectional area) [17, 23, 24, 25]. When the spatial derivative $\frac{\partial u}{\partial x}$ is small, it approaches the wave equation. It is simple, as in the linear case, to introduce a term (proportional to $\frac{\partial u}{\partial t}$) in order to model loss; such a term has little impact on the energy and stability analysis to follow, other than to render conserved quantities dissipated.

### 2.1. First-order System

As discussed in [18], for energetic analysis purposes, it is useful to reduce (1) to a system in the new variables $p$ and $q$ defined by

$$p = \sqrt{\rho}\frac{\partial u}{\partial t} \qquad q = \sqrt{T_0}\frac{\partial u}{\partial x}$$

in which case it can be written as

$$\frac{\partial p}{\partial t} = c_0 G\frac{\partial q}{\partial x} \qquad (2a)$$

$$\frac{\partial q}{\partial t} = c_0\frac{\partial p}{\partial x} \qquad (2b)$$

where we have introduced the quantities $c_0 = \sqrt{T_0/\rho}$ and $G$, as defined by

$$G \triangleq \left(1 + B\int_0^L q^2 dx\right) \qquad (3)$$

with $B \triangleq EA/2LT_0^2$.

### 2.2. Energy Conservation

As also discussed in [18], the Kirchhoff-Carrier equation implies a conservation law, given by

$$E_{KC} = \frac{1}{2}\|p\|^2 + \frac{1}{2}\left(1 + \frac{B}{2}\|q\|^2\right)\|q\|^2 = \text{constant} \qquad (4)$$

where $\|f\| = \left(\int_0^L f^2 dx\right)^{1/2}$ for square-integrable functions $f \in L^2(0, L)$. This further implies the bounds

$$\|p\| \leq \sqrt{2E_{KC}} \qquad \|q\| \leq \sqrt{\frac{-1 + \sqrt{1 + 4BE_{KC}}}{B}} \qquad (5)$$

In other words, the size of the state of the solution is bounded in terms of the initial energy present in the string.

## 3. FOURIER DECOMPOSITION AND A NUMERICAL SCHEME

Under fixed boundary conditions at either end of the string, i.e., $u(0,t) = u(L,t) = 0$, in terms of $p$ and $q$, we must have

$$p(0,t) = p(L,t) = 0 \qquad \left.\frac{\partial q}{\partial x}\right|_{0,t} = \left.\frac{\partial q}{\partial x}\right|_{L,t} = 0$$

These conditions suggest the following sine and cosine decompositions of $p$ and $q$:

$$p(x,t) = \sqrt{\frac{2}{L}}\sum_{m=1}^{\infty} P_m(t)\sin(\pi m x/L) \qquad (6a)$$

$$q(x,t) = \sqrt{\frac{2}{L}}\sum_{m=1}^{\infty} Q_m(t)\cos(\pi m x/L) \qquad (6b)$$

which satisfy the above boundary conditions automatically; for free terminations, one may proceed equally easily. (We note that, although in general, the expression for $q(x,t)$ could contain a DC term $Q_0(t)$, the identification of $q(x,t)$ with $\sqrt{T_0}\frac{\partial u}{\partial x}$ means that $q(x,t)$ will be zero mean for any differentiable initial condition $\left.\frac{\partial u}{\partial x}\right|_{t=0}$.)

Parseval's relation implies that

$$\|P\| = \|p\| \qquad \|Q\| = \|q\|$$

where $\|F\| = \left(\sum_{m=1}^{\infty} F_m^2\right)^{1/2}$ for any square-summable sequence $F_m$ (such as $P_m$ or $Q_m$ above. An equivalent form of the conserved energy (4) is then

$$E_{KC} = \frac{1}{2}\|P\|^2 + \frac{1}{2}\left(1 + \frac{B}{2}\|Q\|^2\right)\|Q\|^2 = \text{constant} \qquad (7)$$

Upon substituting the expressions (6) into system (2), one obtains

$$\frac{dP_m}{dt} = -c_0 G\frac{\pi m}{L}Q_m \qquad (8a)$$

$$\frac{dQ_m}{dt} = c_0\frac{\pi m}{L}P_m \qquad (8b)$$

for $m = 1, 2, \ldots$ which is an infinite set of coupled ordinary differential equations, with the coupling occurring through $G = 1 + B\|Q\|^2$. Such a Fourier decomposition for the Kirchhoff-Carrier equation was analyzed some time ago by Dickey [26].

### 3.1. Time Discretization

In order to numerically integrate the system (8), there are two approximations which must be made. First, we truncate the Fourier series representation of $p$ and $q$ to $M$ terms. It is useful to introduce vectors containing the first $M$ components, namely,

$$\mathbf{P}(t) = [P_1, \ldots, P_M]^T \qquad \mathbf{Q}(t) = [Q_1, \ldots, Q_M]^T$$

Then, we approximate $d\mathbf{P}/dt$, and $d\mathbf{Q}/dt$ as

$$\left.\frac{d\mathbf{P}}{dt}\right|_{t=(n-\frac{1}{2})k} \approx \frac{1}{k}\left(\mathbf{P}^n - \mathbf{P}^{n-1}\right)$$

$$\left.\frac{d\mathbf{Q}}{dt}\right|_{t=nk} \approx \frac{1}{k}\left(\mathbf{Q}^{n+\frac{1}{2}} - \mathbf{Q}^{n-\frac{1}{2}}\right)$$

The quantities $\mathbf{P}^n$ and $\mathbf{Q}^{n+\frac{1}{2}}$ are second-order approximations to $\mathbf{P}((n-\frac{1}{2})k)$ and $\mathbf{Q}(nk)$, respectively; $k$ is the time step. Notice, in particular, that the approximations are interleaved, i.e., we calculate $\mathbf{P}^n$ and $\mathbf{Q}^{n+\frac{1}{2}}$ in alternation, at intervals of $k/2$ seconds.

The inner product of two $M$-component expansions $\mathbf{F}$ and $\mathbf{G}$ is defined in the usual way as

$$\langle \mathbf{F}, \mathbf{G}\rangle = \mathbf{F}^T\mathbf{G} = \sum_{m=1}^{M} F_m G_m \qquad (9)$$

and the norm of any expansion $\mathbf{F}$ follows as

$$\|\mathbf{F}\| = \langle \mathbf{F}, \mathbf{F} \rangle^{1/2} \tag{10}$$

### 3.2. An Interleaved Finite Difference Scheme

System (8) then becomes

$$\begin{aligned} \mathbf{P}^n - \mathbf{P}^{n-1} &= -c_0 k \mathcal{G}^{n-\frac{1}{2}} \mathbf{D} \mathbf{Q}^{n-\frac{1}{2}} && \text{(11a)} \\ \mathbf{Q}^{n+\frac{1}{2}} - \mathbf{Q}^{n-\frac{1}{2}} &= c_0 k \mathbf{D} \mathbf{P}^n && \text{(11b)} \end{aligned}$$

Here, $\mathbf{D}$, as defined by

$$\mathbf{D} = \frac{\pi}{L} \mathrm{diag}(1, \ldots, M) \tag{12}$$

is the spatial derivative operator, and $\mathcal{G}^{n-\frac{1}{2}}$ is an approximation to $G$ at time $t = (n - \frac{1}{2})k$, to be specified shortly; in order that the scheme (11) remain second-order accurate, the approximation $\mathcal{G}^{n-\frac{1}{2}}$ should be second-order accurate. In order that the scheme remain explicitly computable, it should also be a function only of $\mathbf{Q}^{n-\frac{1}{2}}$.

### 3.3. Energy Conservation

In order to examine the energetic behavior of system (11), we may proceed in the following way: first, left-multiply (11a) by $\frac{1}{2}\left(\mathbf{P}^n + \mathbf{P}^{n-1}\right)^T$ to get

$$\begin{aligned} \frac{\|\mathbf{P}^n\|^2}{2} - \frac{\|\mathbf{P}^{n-1}\|^2}{2} &= -\frac{c_0 k}{2} \mathcal{G}^{n-\frac{1}{2}} \langle \mathbf{P}^n + \mathbf{P}^{n-1}, \mathbf{D}\mathbf{Q}^{n-\frac{1}{2}} \rangle \\ &= -\frac{c_0 k}{2} \mathcal{G}^{n-\frac{1}{2}} \langle \mathbf{D}\left(\mathbf{P}^n + \mathbf{P}^{n-1}\right), \mathbf{Q}^{n-\frac{1}{2}} \rangle \end{aligned}$$

Noting, from (11b), that

$$c_0 k \mathbf{D}\left(\mathbf{P}^n + \mathbf{P}^{n-1}\right) = \mathbf{Q}^{n+\frac{1}{2}} - \mathbf{Q}^{n-\frac{3}{2}}$$

we then arrive at

$$\frac{\|\mathbf{P}^n\|^2}{2} - \frac{\|\mathbf{P}^{n-1}\|^2}{2} = -\frac{\mathcal{G}^{n-\frac{1}{2}}}{2}\left(\langle \mathbf{Q}^{n+\frac{1}{2}}, \mathbf{Q}^{n-\frac{1}{2}} \rangle - \langle \mathbf{Q}^{n-\frac{1}{2}}, \mathbf{Q}^{n-\frac{3}{2}} \rangle\right)$$

At this point, we may make the following choice for $\mathcal{G}^{n-\frac{1}{2}}$, namely,

$$\mathcal{G}^{n-\frac{1}{2}} = 1 + \frac{B}{2}\left(\langle \mathbf{Q}^{n+\frac{1}{2}}, \mathbf{Q}^{n-\frac{1}{2}} \rangle + \langle \mathbf{Q}^{n-\frac{1}{2}}, \mathbf{Q}^{n-\frac{3}{2}} \rangle\right) \tag{13}$$

which is consistent with definition (3), and second order accurate. This then yields

$$\begin{aligned} \frac{\|\mathbf{P}^n\|^2}{2} - \frac{\|\mathbf{P}^{n-1}\|^2}{2} &= -\frac{1}{2}\left(\langle \mathbf{Q}^{n+\frac{1}{2}}, \mathbf{Q}^{n-\frac{1}{2}} \rangle - \langle \mathbf{Q}^{n-\frac{1}{2}}, \mathbf{Q}^{n-\frac{3}{2}} \rangle\right) \\ &\quad -\frac{B}{4}\left(\langle \mathbf{Q}^{n+\frac{1}{2}}, \mathbf{Q}^{n-\frac{1}{2}} \rangle^2 - \langle \mathbf{Q}^{n-\frac{1}{2}}, \mathbf{Q}^{n-\frac{3}{2}} \rangle^2\right) \end{aligned}$$

from which we can extract a conserved quantity $\mathcal{E}^n_{KC}$, defined by

$$\mathcal{E}^n_{KC} = \frac{1}{2}\left(\|\mathbf{P}^n\|^2 + \langle \mathbf{Q}^{n+\frac{1}{2}}, \mathbf{Q}^{n-\frac{1}{2}} \rangle + \frac{B}{2}\langle \mathbf{Q}^{n+\frac{1}{2}}, \mathbf{Q}^{n-\frac{1}{2}} \rangle^2\right) \tag{14}$$

which is similar to the energy definition (4), but which is not necessarily positive. The determination of conditions on its positivity (so that it may be used as a numerical stability guarantee) follows in the next Section.

We also note that from definition (13), it would appear that $\mathcal{G}^{n-\frac{1}{2}}$ is dependent on $\mathbf{Q}^{n+\frac{1}{2}}$, thus rendering our difference scheme implicit. It is simple to show, however, that $\mathcal{G}^{n-\frac{1}{2}}$ may be rewritten as

$$\mathcal{G}^{n-\frac{1}{2}} = \frac{1 + B\|\mathbf{Q}^{n-\frac{1}{2}}\|^2}{1 + \frac{Bc_0^2k^2}{2}\|\mathbf{D}\mathbf{Q}^{n-\frac{1}{2}}\|^2} \tag{15}$$

Notice, in particular, that a choice of $\mathcal{G}^{n-\frac{1}{2}} = 1 + B\|\mathbf{Q}^{n-\frac{1}{2}}\|^2$, perhaps the most straightforward choice, does not lead to a simple energy conservation property.

### 3.4. Numerical Stability

The conserved quantity $\mathcal{E}^n_{KC}$ given by (14), unlike its continuous-time counterpart (7), is not necessarily positive. In this Section, we find the conditions under which it is positive, in which case it can be used to bound the size of the calculated solution, thus serving as a numerical stability guarantee. We first rewrite (14), using (11b), as

$$\begin{aligned} \mathcal{E}^n_{KC} &= \frac{1}{2}\left(\|\mathbf{P}^n\|^2 + \|\mathbf{Q}^{n+\frac{1}{2}}\|^2 - c_0 k \langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle\right) \\ &\quad + \frac{B}{4}\left(\|\mathbf{Q}^{n+\frac{1}{2}}\|^2 - c_0 k \langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle\right)^2 \end{aligned}$$

Examine now the term $\langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle$ in the above expression. From the Cauchy-Schwarz inequality [27], we clearly have

$$\langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle \leq \|\mathbf{D}\mathbf{P}^n\| \cdot \|\mathbf{Q}^{n+\frac{1}{2}}\|$$

and, furthermore,

$$\langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle \leq |||\mathbf{D}||| \cdot \|\mathbf{P}^n\| \cdot \|\mathbf{Q}^{n+\frac{1}{2}}\|$$

where $|||\mathbf{D}|||$ is the induced matrix 2-norm of $\mathbf{D}$ [27]; as $\mathbf{D}$ is simply a scaled diagonal matrix, as given by (12), we have $|||\mathbf{D}||| = \frac{\pi M}{L}$, and thus

$$\langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle \leq \frac{\pi M}{L}\|\mathbf{P}^n\| \cdot \|\mathbf{Q}^{n+\frac{1}{2}}\|$$

It then follows that

$$\begin{aligned} \mathcal{E}^n_{KC} &\geq \frac{1}{2}\left(\|\mathbf{P}^n\| - \frac{c_0 k \pi M}{2L}\|\mathbf{Q}^{n+\frac{1}{2}}\|\right)^2 \\ &\quad + \frac{1}{2}(1 - (\frac{c_0 k \pi M}{2L})^2)\|\mathbf{Q}^{n+\frac{1}{2}}\|^2 \\ &\quad + \frac{B}{4}\left(\|\mathbf{Q}^{n+\frac{1}{2}}\|^2 - c_0 k \langle \mathbf{D}\mathbf{P}^n, \mathbf{Q}^{n+\frac{1}{2}} \rangle\right)^2 \end{aligned}$$

The quantity on the right-hand side of the above inequality is clearly positive if

$$k \leq \frac{2L}{c_0 \pi M} \tag{16}$$

If this condition is satisfied (and notice that it does not depend in any way on values of the solution), then we further have that

$$\|\mathbf{Q}^{n+\frac{1}{2}}\| \leq \sqrt{\frac{2\mathcal{E}_{KC}}{1 - (\frac{c_0 k \pi M}{2L})^2}} \tag{17}$$

Thus the norm of $\mathbf{Q}^{n+\frac{1}{2}}$ is bounded in terms of the energy $\mathcal{E}_{KC}$, which remains constant. An identical bound can be found for $\|\mathbf{P}^n\|$. (We note that it should be possible to find tighter bounds through further analysis.)

This bound implies a further bound on $\mathcal{G}^{n-\frac{1}{2}}$, as defined by (15), namely

$$\mathcal{G}^{n-\frac{1}{2}} \leq 1 + \frac{2B\mathcal{E}_{KC}}{1 - (\frac{c_0 k \pi M}{2L})^2} \tag{18}$$

which holds for all $n$.

### 3.5. Oscillatory Behavior

We have found, above, a condition for numerical stability; it is not, however, sufficient to ensure that our calculated solution is acceptable from a physical point of view, as numerical oscillatory behavior may be present, even if the solution is stable. To this end, we rewrite system (11) in state-space form as

$$\begin{bmatrix} \mathbf{P}^n \\ \mathbf{Q}^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_M & -c_0 k \mathcal{G}^{n-\frac{1}{2}} \mathbf{D} \\ c_0 k \mathbf{D} & \mathbf{I}_M - c_0^2 k^2 \mathcal{G}^{n-\frac{1}{2}} \mathbf{D}^2 \end{bmatrix} \begin{bmatrix} \mathbf{P}^{n-1} \\ \mathbf{Q}^{n-\frac{1}{2}} \end{bmatrix}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. The $2M$ eigenvalues $\lambda^n$ of the update matrix (which is dependent on the time index $n$) are given by

$$\lambda^n = -\nu^n \pm \sqrt{(\nu^n)^2 - 1} \qquad \nu^n = -1 + \frac{c_0^2 k^2 \pi^2 m^2 \mathcal{G}^{n-\frac{1}{2}}}{2L^2}$$

for $m = 1, \ldots M$. For $|\nu^n| \leq 1$, or, in other words, if

$$k \leq \frac{2L}{c_0 \pi M \sqrt{\mathcal{G}^{n-\frac{1}{2}}}} \tag{19}$$

then all eigenvalues occur as $M$ complex conjugate pairs, of unit magnitude; if, however, $\nu^n > 1$, some eigenvalues are both real and negative, and in particular, one will be of magnitude greater than one. We thus expect, in this case, to find sign-flipping (accompanied by amplification) occurring in the modes whose eigenvalues violate this condition. It is thus important to ensure that this does not occur. Using (18) and (19), we may find another bound on $k$,

$$k \leq \frac{2L}{c_0 \pi M} \sqrt{1 + B\mathcal{E}_{KC} - \sqrt{(1 + B\mathcal{E}_{KC})^2 - 1}}$$

which is expressed in terms of the initial energy $\mathcal{E}_{KC}$.

We again emphasize that as long as the stability condition (16) is satisfied, the solution must remain numerically stable, i.e., the computed values of the solution may be bounded by (17). The amplification of oscillatory modes, discussed above, is a different phenomenon; essentially, energy is drawn from other modes into the oscillatory ones (typically high frequency), leaving the total energy constant. The buildup of this spurious energy must be bounded in terms of the total energy.

### 3.6. Implementation Details

The algorithm itself, described by (11), operates entirely on the sinusoidal expansion coefficients. The most important practical consideration is the transformation between the coefficients of the sinusoidal expansions of $p$ and $q$ and the physical solution; this can be done rather simply using Fourier transforms, though it must be kept in mind that the expansions $\mathbf{P}$ and $\mathbf{Q}$ are sine and cosine series, not general Fourier series expansions of functions over an interval of length $L$. Also, due to the incorporation of boundary conditions into the series, we have chosen to expand into sinusoids of wavelengths $2L/m$, for $m = 1, \ldots, M$. For example, for a sequence $p_i^n$, for $i = 0, \ldots M-1$, representing some approximation to $p$, (here, $p_0$ is constrained to be zero by the boundary conditions), we can generate the expansion coefficients by taking the Fourier transform of the sequence

$$[p_0, p_1, \ldots, p_{M-1}, 0, -p_{M-1}, \ldots, -p_1]$$

and taking the imaginary parts of the first $M$ values (in other words, we expand $p_i$ to an odd sequence of length $2M$ samples). Similarly, a set of values representing $q$ can be expanded to a length $2M$ even sequence, and the real parts of the first $M$ values will represent the cosine series coefficients. Simplifications are certainly possible, given the various symmetries of the sequences; we do not enter into the details here. It is important to note that if one is merely interested in viewing the state of the string after an elapsed time, there is no need to perform a Fourier transform until this instant. Otherwise, for synthesis purposes, a Fourier transform must be taken at each sampling instant. For this reason, it will probably be of interest to constrain $M$ to be a power of two, so that the FFT algorithm may be used.

Another question is that of initialization. As mentioned previously, the Kirchhoff-Carrier system (1) requires two initial conditions, $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$. Scheme (11) also requires two initial conditions, $\mathbf{P}^0$ and $\mathbf{Q}^{\frac{1}{2}}$; due to interleaving, they do not occur at the same time instant. $\mathbf{P}^0$ may be rather simply set as $\sqrt{\rho}$ times the Fourier expansion coefficients of $\frac{\partial u}{\partial t}$ at $t = 0$, but $\mathbf{Q}^{\frac{1}{2}}$ requires a more delicate treatment. The simplest means of proceeding is simply to find the set $\mathbf{Q}^{\frac{1}{2}}$ to be $\sqrt{T_0}$ times the cosine expansion coefficients of the spatial derivative $\frac{\partial u}{\partial x}$ at time $t = 0$ (perhaps through spectral differentiation of the initial displacement $u(x,0)$, or some other means). This means that we will have to accept a first-order error, due to the offset in the initial data. Alternatively, it is possible to develop a special scheme to be used only once, for initialization purposes [19].

## 4. NUMERICAL SIMULATIONS

In this Section we present a few simulation results, using the algorithm (11). In Figure 1 we show snapshots of the time evolution of a steel string, under center-plucked conditions, for a variety of amplitudes (parameters as given in the caption to the figure). For low amplitudes (left column), the motion is, as expected, very similar to what one would expect from the wave equation, i.e., a triangular initial displacement gives rise to simple propagating "corners." As the amplitude is increased (second and third columns), the truncated triangular shape becomes progressively more distorted; notice also that the corners propagate more rapidly for higher amplitudes. This is in line with what we expect of nonlinear plucked excitations, i.e., there should be an increased wave speed, leading to an increased perceived frequency of oscillation (if it can be called that). This point is made more clearly in the upper row of Figure 2, where we show the displacement of the string center as a function of time, for the same plucking conditions as in the columns of Figure 1. As a test of the energy conservation properties of the algorithm, we have also plotted the difference between string energy and the initial energy, normalized by the initial energy, as a function of time, in the bottom row of Figure 2, for the same set of excitations. Notice that the energy error is zero, to machine precision.

## 5. CONCLUSIONS

We have discussed here an extension of Fourier techniques to a nonlinear model of string vibration. The algorithm presented in this paper is a special case of what is known as a *spectral method* [28, 29]. In short, the spatial derivative operator has been approximated by frequency domain multiplication, and is thus exact (i.e.,

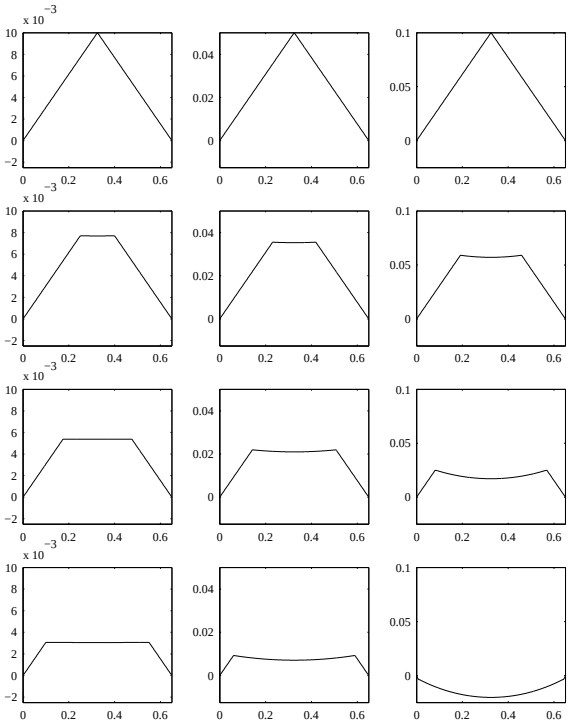

Figure 1: *Time evolution of the profile of a string described by the Kirchhoff-Carrier equation, under the application of the energy-conserving difference scheme* (11)*; the string is of length* $0.65$ m*, made of steel (of linear density* $\rho = 6 \times 10^{-4}$ kg/m *and with Young's modulus* $E = 2 \times 10^{11}$N/m$^2$*), of cross-sectional area* $A = 3.6 \times 10^{-8}$m$^2$*, under tension* $T_0 = 120$N*, and is subject to a triangular (center-plucked) initial condition. Snapshots are taken at times* $t = 0$ s, $t = 0.000166$ s, $t = 0.000333$ s *and* $t = 0.0005$ s*, for a variety of initial amplitudes: (a)* $0.01$ m*, (b)* $0.05$ m*, and (c)* $0.1$ m.

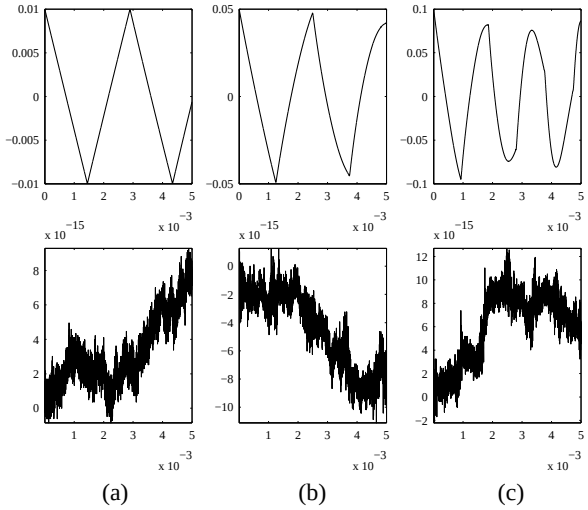

Figure 2: *Time waveforms and energy error, for the string of parameters as given in the caption to Figure 1, for triangular (center-plucked) initial conditions, of amplitudes (a)* $0.01$ m*, (b)* $0.05$ m*, and (c),* $0.1$ m*. The displacement of the center of the string is plotted as a function of time, over the interval* $t \in [0, 0.005]$s*. In each case, the difference between the current energy* $\mathcal{E}_{KC}$ *and the initial value of this energy, divided by the initial energy is plotted as a function of time over the same interval.*

has no truncation error), at least over the range of solutions which may be expressed in terms of a fixed number of sinusoidal components. In particular, the spatial accuracy is far greater than that of a finite difference scheme for the same system [18]. Time discretization limits temporal accuracy to second order; it would, of course, be possible to use higher-order accurate methods (perhaps of the Runge-Kutta variety) for time integration, but we have chosen a simple interleaved scheme in order to highlight the special energy conservation property which is crucial for stability analysis. Indeed, a robust stability guarantee is of paramount importance for sound synthesis, especially for real-time applications; this can be difficult to achieve for nonlinear systems. We have also shown a means of controlling parasitic oscillations, which, interestingly, arise independently of numerical stability.

## 6. REFERENCES


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