# ENERGY-STABLE MODELLING OF CONTACTING MODAL OBJECTS WITH PIECE-WISE LINEAR INTERACTION FORCE

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## ABSTRACT

In discrete-time digital models of contact of vibrating objects stability and therefore control over system energy is an important issue. While numerical approximation is problematic in this context digital algorithms may meat this challenge when based on exact mathematical solution of the underlying equation. The latter may generally be possible under certain conditions of linearity. While a system of contacting solid objects is non-linear by definition, piece-wise linear models may be used. Here however the aspect of "switching" between different linear phases is crucial. An approach is presented for exact preservation of system energy when passing between different phases of contact. One basic principle used may be pictured as inserting appropriate ideal, massless and perfectly stiff, "connection rods" at discrete moments of phase switching. Theoretic foundations are introduced and the general technique is explained and tested at two simple examples.

## 1. INTRODUCTION AND BACKGROUND

The modelling of contact of solid objects is a prominent challenge in the fields of sound synthesis and virtual reality (e.g. [1]). For describing the inner laws and attributes of solid objects the modal approach [4][2][3] has been very successful [1] [5] [6] [7] which can be applied under certain conditions of linearity described below. Contact has been modelled in various ways, mostly using force profiles chosen a priori (e.g. [1]), such that the resulting behaviour of the affected object(s) can be seen regarded as a filter with force(s) as input. The derivation of such temporal force profiles is however often very heuristic and there is no clear strategy for changes of different forms of contact. For example is it not a priori clear what the characteristic differences of force profiles in rolling or sliding are. Further on, this approach has strong limitations in realtime interactive scenarios, e.g. when a virtual object passes between phases of rolling and of bouncing.

Other models of contact include laws for the forces of contact in dependence of the configuration of the involved objects. On this basis, numerical simulations may be implemented in which temporal force profiles are computed along with the objects' resonance behaviour (see e.g. [7]). This latter approach has strong potential for realtime interaction, in particular in dynamical situations of continuous or repeated contact, such as in rolling–bouncing interaction [8]. The contact model presented in [7] is used for a variety of sound design tasks (e.g. [9], [10]). A central problem in such scenarios of frequent or continuous contact lies in the stability of the discrete-time algorithms: since these are generally based on numerical approximation of differential equations, control over the exact terms of energy in the system is lost and artifacts such as a falling object bouncing forever under the influence of gravity may occur.

As shortly discussed in the next subsections, numerical artifacts that affect system energy and thus stability can be avoided for linear systems by basing discrete-time algorithms on *exact* mathematical solution rather than numerical *approximation*. A simple argument however shows that a system of two solid objects that interact when in contact and otherwise behave independently can as a whole (over both phases, contact and no contact) *not* be described by a linear model: here, the interaction force acting between both objects would form a linear function of the state vectors of both objects, with value $0$ on an open subset of the global state (vector) space. (One may only look at the system at any configuration where both objects are at some positive distance and note that no interaction force will occur for any object positions which are sufficiently close to this initial configuration.) Such a function must however be identically $0$ *everywhere* as the kernel of a linear function forms a linear subspace. The approach presented in the following allows to guarantee control over the energy of the system in the discrete-time algorithm and thus complete stability also in any situation of repeated or continuous contact. The basic idea is here to apply the modal approach also during contact of both objects, which is possible if the interaction force is governed by a piece-wise linear law. Crucial hereby is the aspect of how to "switch" between different linear phases, for which a simple but satisfactory — for the present goals — solution is presented in section 2.2.

Since the modal description and formalism are at the core of the work, the following two subsection very shortly summarise its main theoretical principle and its practice in the finite-dimensional case.

### 1.1. Modal approach — general principle

Contrary to the impression sometimes created in literature, modal description is not necessarily based on approximation of object behaviour by discrete lumped spatial elements or by some kind of "resonance" filters. Rather, the fundamental underlying principle of expressing a linear operator acting on state vectors of a physical system by means of eigenvectors is based on an exact mathematical theory and may be applied to spatially discrete as well as continuous systems. Starting point is a description of the temporal behaviour of a physical system in the form

$$\dot{\vec{z}}(t) = A\vec{z}(t) + \vec{f}_{\text{ext}}(t) \tag{1}$$

where $\vec{z}(t)$ is the state of the system in the "*state space*" vector space $Z$ — thus *state "vector"* — at time $t$, $\dot{\vec{z}}(t)$ its temporal derivative, and $A$ a linear operator defined on this state space. The demand on $A$ to be linear is central for the term of "modes of a system" to make sense. It is here stressed that the state space may be finite- or also infinite-dimensional.

The core idea of the modal approach is to simplify and solve equation (1) by expressing the state vector $\vec{z}$ in a basis of (generalised) eigenvectors of $A$. To only quickly illustrate this principle one may look at the case of the "homogeneous" form of equation (1), $\vec{f}_{\text{ext}}(t) = \vec{0}$, i.e. under absence of external forces, and assume $\vec{z}(t)$ to lie inside one eigenspace of $A$ to eigenvalue $d$: (1) then reduces to $\dot{\vec{z}}(t) = d \cdot \vec{z}(t)$ (with $d$ *scalar*!), which is readily solved by means of an exponential function (in time): $\vec{z}(t) = e^{dt} \cdot \vec{z}(0)$. While this simple example serves to illustrate the general idea of solving equation (1) by representing the operator $A$ and state vectors $\vec{z}$ in a suitable form, the exact mathematical theory may be highly difficult and abstract (in particular in the case of infinite-dimensional state spaces, see e.g. [11]). For the practical application at scenarios of contact described in this contribution it is however sufficient to understand some main facts for the case of finite-dimensional state spaces, which are shortly summarised in the next subsections.

### 1.2. Finite-dimensional/spatially discrete case

In the case of a finite-dimensional state space the operator $A$ in equation (1) be can represented by (or regarded as) a matrix. In this case the modal approach of finding the generalised eigenvectors of $A$ consists in finding a similarity transformation for $A$ to (e.g.) Jordan canonical form, i.e. of finding a non-singular matrix $V$ such that

$$N := V^{-1}AV \qquad (2)$$

is of Jordan canonical form [12]. The proof that such a transformation exists for any matrix $A$ and techniques how it can be practically derived are results from linear algebra (see e.g. [12]) and numerics []. Without going into further detail it is noted that in most practical cases the Jordan canonical form is diagonal, i.e. $A = VDV^{-1} (\Leftrightarrow D = V^{-1}AV)$, where $D$ is a diagonal matrix.

#### *1.2.1. Stiffness and friction matrices*

The most common practical application of the modal approach is in the situation of a system of second order differential equations of the form

$$M\ddot{\vec{x}}(t) + C\dot{\vec{x}}(t) + K\vec{x}(t) = \vec{f}^{\text{ext}}(t). \qquad (3)$$

The column vector $\vec{x} = (x_1 \ldots x_n)^{\mathrm{T}}$ ("T" denoting matrix transposition) holds the (finite number of) discrete "displacement" variables, $\vec{f}^{\text{ext}}$ is a vector of external forces acting on the system, and $M$, $K$ and $C$ are matrices representing the dependence of forces of inertia, "stiffness" and "friction" on the configuration of the system. The variables $x_1, \ldots, x_n$ and equation (3) most often derive from Newton's laws for an idealised system of lumped masses or from Lagrange equations. They may however also be derived from a spatially continuous system which has first been transformed directly by the modal approach and then simplified to a finite number of "modes". This latter case is the most common in sound generation by "*modal synthesis*". Such a situation of using modal parameters in an already "abstracted" way is also the one mainly aimed at with the approach to modelling contact described in the following (as will become clear in subsection 2.3). Furtheron, the parameters represented by the matrices $M$, $K$ and $C$ are in practice often derived from measurements at mechanical objects.

The inertia matrix $M$ is generally invertible (most often diagonal) and may therefore be omitted by passing from $K$ and $C$ to $M^{-1}K$ and $M^{-1}C$, so that equation (3) becomes

$$\ddot{\vec{x}}(t) + C\dot{\vec{x}}(t) + K\vec{x}(t) = M^{-1}\vec{f}^{\text{ext}}(t). \qquad (4)$$

Often (and this is the case covered in most literature) $K$ and $C$ may be diagonalised simultaneously, which is possible if and only if one of them can be diagonalised and both commute, i.e. $KC = CK$ [12]. We here however need to account for the general case where $K$ and $C$ do not commute. This is handled by introducing the state vector $\vec{z} := \begin{pmatrix} \vec{x} \\ \dot{\vec{x}} \end{pmatrix}$. Defining $A := \begin{pmatrix} O & E \\ -K & -C \end{pmatrix}$, where $E$ denotes the identity matrix (of dimension equal to $K$ and $C$), equation (4) then takes the form

$$\dot{\vec{z}}(t) = A\vec{z}(t) + \begin{pmatrix} \vec{0} \\ M^{-1}\vec{f}_{\text{ext}}(t) \end{pmatrix}, \qquad (5)$$

which is equivalent to (1) (finite-dimensional).

#### *1.2.2. Transition matrix*

The most simple way to see the possibility of discrete-time simulation of the temporal behaviour of a finite-dimensional homogeneous system $\dot{\vec{z}}(t) = A\vec{z}(t)$ is by noting that its solution can be directly given in the form

$$\vec{z}(t) = e^{tA}\vec{z}(0). \qquad (6)$$

The exponential function of a matrix, $e^M$, may, e.g., be defined by the series expression of the exponential function. It is seen that, just as the value of $\vec{z}$ at $t = t_\Delta$ can be determined by multiplying $\vec{z}(0)$ with $e^{t_\Delta A}$ (equation (6)), in the same way $\vec{z}(t + t_\Delta) = e^{t_\Delta A}\vec{z}(t)$. The matrix $e^{t_\Delta A}$ is therefore also called "*(state) transition matrix*" (to the time step $t_\Delta$) as it allows to pass from any one temporal state vector to the one at the moment $t_\Delta$ later [2]. $e^{t_\Delta A}$ can not easily be determined by means of the series expression, but it is remarked that for $A = VDV^{-1}$ as in equation (2) we have

$$e^{t_\Delta A} = Ve^{t_\Delta D}V^{-1}. \qquad (7)$$

The matrix $e^{t_\Delta D}$ however can easily be seen to be of diagonal form again, with entries given by the *scalar* exponential function of the entries of $D$. Diagonalisation of $A$ is thus also a technique to determine the transition matrix and we have closed the cycle, turning back to the modal approach.

In practice one will often not use the transition matrix $e^{t_\Delta A}$ referring to spatial coordinates (or whatever coordinates have initially been used to formulate the model, e.g. Lagrange equations. . . ). Rather, by permanently using the state vector $\vec{z}_{\text{mod}}(t) := V^{-1}\vec{z}(t)$ one will work with the transition matrix $e^{t_\Delta D}$ which is, as noted, of diagonal form and thus computationally much more effective. This change of coordinates is particularly convenient when not the complete spatial behaviour of the modelled system needs to be known: in sound synthesis it is often sufficient to know the temporal movement of one or a few "pickup point(s)" of a vibrating structure, simular to the situation of an electro-magnetic pickup of an electric guitar or piano.

## 2. MODELLING CONTACT BY MODAL DESCRIPTION

Returning to the general abstract formulation of subsection 1.1 in order to explain our approach to modelling contact, take two systems of the described type of equation (1)

$$\dot{\vec{z}}_1(t) = A_1\vec{z}_1(t) + \vec{f}_{1\text{ext}} \text{ and } \dot{\vec{z}}_2(t) = A_2\vec{z}_2(t) + \vec{f}_{2\text{ext}} \quad (8)$$

interacting by means of some interaction force $\vec{f}(\vec{z}_1, \vec{z}_2)$ which in this formulation makes part of the external forces $\vec{f}_1$ and $\vec{f}_2$. For simplicity we assume for now that no other external forces exist and that the "actio–reactio" principle holds such that (with no restrictions on generality) $\vec{f}_1 = \vec{f}$ and $\vec{f}_2 = -\vec{f}$. If then $\vec{f}$ is given as a *linear* function of $\vec{z}_1$ and $\vec{z}_2$ the whole system of both masses in contact may again be written in the form $\dot{\vec{z}}(t) = A\vec{z}(t)$ with a *linear* operator $A$ on the space $Z_1 \times Z_2$, "built from" $A_1$, $A_2$ and the linear expression of $\vec{f}$. While the clean mathematical formulation of this process in the general case is rather tedious (although not complicated), the idea is demonstrated and realized in a concrete example in the following subsections.

### 2.1. Contact: point-mass – finite-dimensional system

We return to the finite-dimensional system of subsection 1.2 described by equation (4). This object shall be struck by a "hammer", which, in order to keep the overall system possibly simple and demonstrate the general idea, is assumed to behave as a point-mass free to move along one spatial direction. In the contact-less "free" configuration the hammer is thus described by the scalar position variable $x_{n+1}$ (the reason for choosing the subscript "$_{n+1}$" will become clear in a minute) and behaving according to $m_h\ddot{x}_{n+1} = f$, with $m_h$ its mass. Again for simplicity we assume that no external forces other than the hammer–object interaction are present such that $\vec{f} = 0$ (thus also $f = 0$) when both objects are not in contact. The hammer may touch the object at one specific point with index $l_{\text{con}}$ (moving along the same spatial coordinate as this "contact mass") and when such contact occurs the interaction shall be modelled as a massless damped-spring connection with stiffness $k_{\text{con}}$ and friction constant $c_{\text{con}}$. Summing up, the contact force $f$ is thus given by the following term:

$$f = \begin{cases} k_{\text{con}}(x_{l_{\text{con}}} - x_{n+1}) + c_{\text{con}}(\dot{x}_{l_{\text{con}}} - \dot{x}_{n+1}), \\ \qquad\qquad \text{for } x_{n+1} < x_{l_{\text{con}}} \\ 0, \qquad\qquad \text{otherwise.} \end{cases} \quad (9)$$

This force term is analogous to the one used in [7] except that it is linear. It will be seen in the following that the restriction of this choice allows big advantages in solving and implementing the system.

Remembering that the force acting on $m_{l_{\text{con}}}$ according to the actio–reactio principle is $-f$, the dynamical equations for the two affected masses, i.e. the "hammer" and the mass $m_{l_{\text{con}}}$, during contact are

$$\begin{aligned} \ddot{x}_{n+1} &= \frac{f}{m_h} \\ &= \tfrac{k_{\text{con}}}{m_h}x_{l_{\text{con}}} - \tfrac{k_{\text{con}}}{m_h}x_{n+1} + \tfrac{c_{\text{con}}}{m_h}\dot{x}_{l_{\text{con}}} - \tfrac{c_{\text{con}}}{m_h}\dot{x}_{n+1} \end{aligned} \quad (10)$$

and

$$\begin{aligned} \ddot{x}_{l_{\text{con}}} &= f^{(\text{int})}_{l_{\text{con}}} + \frac{f^{(\text{ext})}_{l_{\text{con}}}}{m_{l_{\text{con}}}} = f^{(\text{int})}_{l_{\text{con}}} - \frac{f}{m_{l_{\text{con}}}} = f^{(\text{int})}_{l_{\text{con}}} \\ &+ \tfrac{k_{\text{con}}}{m_{l_{\text{con}}}}x_{n+1} - \tfrac{k_{\text{con}}}{m_{l_{\text{con}}}}x_{l_{\text{con}}} + \tfrac{c_{\text{con}}}{m_{l_{\text{con}}}}\dot{x}_{n+1} - \tfrac{c_{\text{con}}}{m_{l_{\text{con}}}}\dot{x}_{l_{\text{con}}}, \end{aligned} \quad (11)$$

where $f^{(\text{int})}_{l_{\text{con}}}$ is the internal force acting on $m_{l_{\text{con}}}$ inside the object (described by equation (4) in subsection 1.2). We combine the vector of the positions of the $n$ masses of the object of subsection 1.2 with the position of the hammer $x_{n+1}$ into a new position vector $\acute{\vec{x}} := (x_1 \ldots x_{n+1})^{\mathrm{T}}$. The influence of the contact force described by equations (10) and (11) on the whole system can then be written in the form

$$\ddot{\acute{\vec{x}}}(t) = -K_{\text{con}}\acute{\vec{x}}(t) - C_{\text{con}}\dot{\acute{\vec{x}}}(t) + \begin{pmatrix} \vec{f}^{(\text{int})}(t) \\ 0 \end{pmatrix}, \quad (12)$$

with

$$K_{\text{con}} := \begin{pmatrix} 0\ldots & & & & 0 \\ \vdots\ddots & & & & \\ & \vdots & \frac{k_{\text{con}}}{m} & 0\ldots0 & -\frac{k_{\text{con}}}{m} \\ & & 0 & 0\ldots0 & 0 \\ & & \vdots & \ddots & \vdots \\ & & 0 & 0\ldots0 & 0 \\ 0\ldots0 & & -\frac{k_{\text{con}}}{m_h} & 0\ldots0 & \frac{k_{\text{con}}}{m_h} \end{pmatrix} \begin{matrix} \\ \\ \leftarrow l_{\text{con}}\text{th row} \\ \\ \\ \\ \\ \end{matrix} \quad (13)$$

and $C_{\text{con}}$ analogous to $K_{\text{con}}$, $k_{\text{con}}$ replaced by $c_{\text{con}}$. $\vec{f}^{(\text{int})}$ is here the vector of all internal forces inside the object, described just by the matrices $K$ and $C$ in subsection 1.2, equation (4). We may therefore combine the matrices $K$ and $C$ with $K_{\text{con}}$ and $C_{\text{con}}$ (note that the dimensionality of the "$_{\text{con}}$" matrices is by one higher than that of $K$ and $C$) into

$$\acute{K} := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} + K_{\text{con}} \text{ and } \acute{C} := \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + C_{\text{con}} \quad (14)$$

and write the final equation that describes the behaviour of the entire system during contact as

$$\ddot{\acute{\vec{x}}}(t) + \acute{K}\acute{\vec{x}}(t) + \acute{C}\dot{\acute{\vec{x}}}(t) = \vec{0}, \quad (15)$$

again assuming absence of any additional external forces (thus the $\vec{0}$-vector on the right side). Equation (15) is exactly of the same form as equation (4) and may be handled in the exact same way, using a state vector $\acute{\vec{z}} := \begin{pmatrix} \acute{\vec{x}} \\ \dot{\acute{\vec{x}}} \end{pmatrix}$ and a matrix $\acute{A} := \begin{pmatrix} O & E \\ -\acute{K} & -\acute{C} \end{pmatrix}$.

It is noted that $\acute{K}$ and $\acute{C}$ generally do *not* commute (i.e. $\acute{K}\acute{C} \neq \acute{C}\acute{K}$) — e.g. when $K$ and $C$ commute, $c_{\text{con}} = 0$, and $m \neq m_h$. The more general approach of introducing the state vector $\acute{\vec{z}}$ and the matrix $\acute{A}$, as described in subsection 1.2, is thus necessary: it is *not* sufficient to bring $\acute{K}$ (or $\acute{C}$) to Jordan canonical form to solve the above equation (15). It is pointed out that the procedure just presented is independent of the concrete matrices $K$ and $C$ describing the object. It is identically applicable for any matrices $K$ and $C$ describing different geometries or physical systems. Furthermore, the method can also be applied if the second object involved, here the "hammer", is not simply a point-mass, but described in the same general form of equation (3). Finally it is again noted that the general idea is also applicable for infinite-dimensional state spaces, although this case may typically not allow for an analytical solution or be technically much more involved.

### 2.2. Discrete-time realization of contact condition

In principle, the temporal behaviour of a system of two contacting objects that interact according equation (8) can now be simulated by using transition matrices $e^{A_1 t_\triangle}$ and $e^{A_2 t_\triangle}$ during phases of no contact and a transition matrix $e^{\acute{A} t_\triangle}$ with $\acute{A}$ as in the previous subsection during contact (as explained in subsection 1.2.2). The time-step $t_\triangle$ accords to some chosen "sample rate". In practice one will use transformations to suitable "modal coordinates" in both phases, such that the transition matrix turns out to be of some possibly simple form, as shortly explained in subsection 1.2.2. The application of such transformations does not introduce any difficulties and does not change the general described approach and is thus not discussed in detail here. A crucial question however is, when to "switch" between phases of contact and no contact, i.e. when to use either transition matrix (resp. pair of transition matrices): In theory one would need to know in advance when both objects reach just distance $0$, calculate the state vector (or vectors) in that exact moment, decide from this state if the following phase will be one of contact or no contact and finally continue with the simulation — and so on. . . In practice we can *not* predict these moments of switching phases at distance $0$ *exactly*, only approximately, and thereby have to account for questions of stability of the overall discrete-time algorithm as well as costs of computation. A possible solution to this challenge is presented in the following, once again at the example of the previous subsection (2.1).

We start with hammer and object in well defined initial states at $t = 0$. Without loss of generality these shall be such that both objects are not in contact, i.e. of positive distance. From these initial states we compute both states at $t = t_\triangle$, using the transition matrices for the "no contact" phase. As long as both objects stay at positive distance, this state update is exact, since it is based on an analytical solution, *not* some approximation (compare subsection 1.2.2). In an audio application $t_\triangle$ will typically be according to some chosen constant sample rate. Repeating this step-wise state update the distance of both objects will at some point become negative for the first time. The last update step before will then have been wrong, as contact must occur sometime before the end of the time step and a switch to the "contact phase" would have been in place. However, a use of the transition matrix $e^{\acute{A} t_\triangle}$ for the "contact phase" for the last update step (the one just *before* the first occurrence of negative distance) would be wrong as well, since both objects are at positive distance at the end of the previous time step. To make things worse, both errors — of using the transition matrix for the "contact phase" *too early* or *too late* — lead to an erroneous increase of energy in the whole system, since both amount to ad-hoc insertion of the "contact spring" in a stretched state, while the overall kinetic energy in the system is not affected by switching from "no contact" to "contact". For single short impact events this increase of energy may probably be controlled to remain small enough not to disturb the global behaviour of interest, but for situations of repeated contact, such as when an object bounces back under the influence of gravity, the long-term stability of the system or the "macroscopic quality" of its temporal behaviour might be affected. Before suggesting a solution to this problem it is remarked that the situation is just opposite when switching from "contact" to "no contact": in this situation the artifact of inexact, only approximate, knowledge of the time when contact ends always leads to a decrease of the overall energy of the system, since then a stretched "contact spring" is cut ad-hoc, being it "too early" or "too late".

The approach used by the author consists in adding at each contact some spatial offset to the distance of both objects of just such amount that distance zero occurs exactly at the beginning of one update step. Physically, this may be seen as introducing an additional massless, perfectly stiff connecting element between one object and the "contact spring". This geometrical manipulation assures that system energy is *exactly* preserved at the switch from "no contact" to "contact". From the moment when zero distance is reached, which now falls exactly on the beginning of one time step, the system is updated by means of the transition matrix for the "contact phase", until a positive distance (under consideration of the "*offset element*") is first found again. We then switch back to "no contact", hereby loosing some system energy as a result of the discrete-time artifact of "cutting" the stretched contact spring. In the whole however, discrete-time artifacts *never* lead to an increase of energy, such that the system is always stable. Effects such as an object bouncing forever under the influence of gravity can not occur. Of course the offset connection element has to be adapted at each occurring contact which might seem a rather strong geometric manipulation. It is however harmless in our context since unpredictable dynamical changes in surface structures form part of the modelled situations we are mostly interested in, such as in rolling interaction. Also, the level of these offset values may be well estimated and controlled a priori by the choice of the sample rate in dependence of initial system energy. In practical implementation examples, values of the *offset element* were in an order of $10^{-5}$ smaller than the amplitudes of the audible object vibrations.

Figure 1 shows phases of an impact computed by the algorithm just described, of a free mass with "contact spring" and a "string" approximated by $50$ lumped masses and handled by the modal formalism. Parameters of elasticity and friction of the string are such that the lowest modal frequency is around $350$ Hz. The algorithm runs in realtime at a sample rate of $44100$ Hz on an average notebook (even with more than ca. $100$ masses resp. modes, depending on the exact hardware). Of course, a one-dimensional string will in practice rather be modelled by using waveguide techniques [13] which are computationally more economic (although more demanding in terms of processing memory, a fact that may be relevant when working with certain specific dsp hardware). The example is here just used to demonstrate the general practicability of the presented approach. The phenomenon of wave propagation that may be observed in figure 1 is thereby a good confirmation of the validity and exactness of all involved operations (diagonalisation, discrete-time transition matrix, impact algorithm. . . ) since it does in this approach *not* follow for reasons of the computational structure itself, as in the case of digital waveguides, but occurs as a result of superposition of the computed "modes", i.e. eigenvectors and their time-dependent multiplicities. It is noted that the approach may be applied to more complex structures as well, such as membranes, whose modelling in terms of waveguides is not as straightforward as in the case of a one-dimensional string. In practice one will often gain modal data not from spatially lumped models but analytically or from mechanical measurements and then reduce the system to finite dimensions. The application of the presented technique of contact modelling in the case of such a use of modal data in a somewhat more abstract way is demonstrated in the next subsection.

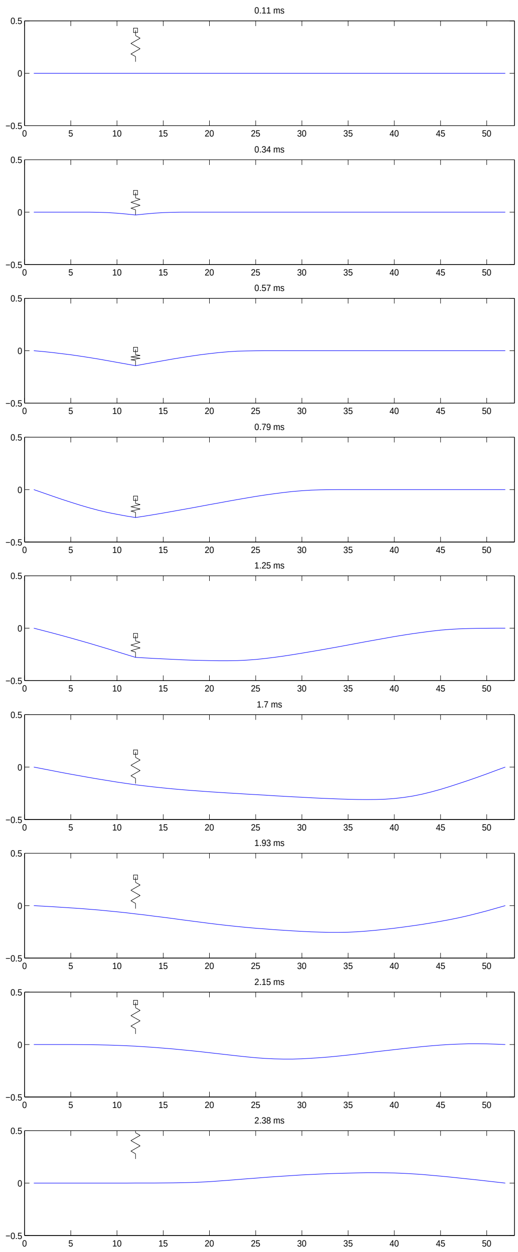


Figure 1: *Snapshots of a string struck by a free mass "hammer".*

### 2.3. Contact of "abstract" finite-dimensional modal objects

In most practical cases of sound generation by modal synthesis objects are described in a "more abstract" way, starting immediately with the modal parameters of eigenfrequencies and decay times (compare e.g. [1][7]). These modal parameters are then used independently of their initial origin. They may have been derived by diagonalisation of a finite point-mass system as described in subsection 1.2, from analytical solution of spatially distributed systems (with therefore infinite-dimensional state space), or, most often, from measurements of "real" mechanical or electrical systems. Examples of infinite-dimensional systems for which an analytical derivation of according eigenvectors and -values is possible are certain one-dimensional systems with homogeneous mass distribution such as beams or strings (for which eigenvalues are commonly known to be the integer multiples of one "fundamental frequency") and two- or three-dimensional systems with strong symmetries such as rectangluar or circular membranes or plates [14]. In practical implementations the number of modes is necessarily finite, which generally demands a simplification of the system to be modelled, being it a partial differential equation or a real distributed object. Strategies and guidelines have been studied to perform such reduction of the number of modes in a way such that consequences in terms of auditory perception are possibly small [15]. In particular, "overdamped" and "free-body" modes [2] can generally be neglected for sound synthesis such that finally, an "abstract" modal object consists of a finite number of pairs of modal frequencies and according decay times (compare [7] [1]). In matrix notation as in subsection 1.2 $K$ and $C$ are then of diagonal form. Analog to equation (4) the temporal behaviour of an abstract object in "modal coordinates" is described by the equation

$$\ddot{\vec{x}}_{\text{mod}}(t) + D_c\dot{\vec{x}}_{\text{mod}}(t) + D_k\vec{x}_{\text{mod}}(t) = \vec{f}^{\text{ext}}_{\text{mod}}(t). \qquad (16)$$

$D_k$ and $D_c$ are here the diagonal matrices corresponding to $K$ and $C$ and $\vec{f}^{\text{ext}}_{\text{mod}}$ is the vector of the sum of external forces acting on the object *expressed in modal coordinates*.

The connection between the modal coordinates of the abstract modal object and spatial coordinates is finally given by means of weighting factors associated to each mode at each potential point of contact. These weighting factors form a reduction of the system of eigenvectors — in the finite-dimensional case of subsection 1.2 the matrix $V$ (resp. $V^{-1}$) of eigenvectors. For one single (scalar) force $f(t)$ acting on the modal object at one point of contact, we have for the external force vector in modal coordinates of equation (16):

$$\vec{f}^{\text{ext}}_{\text{mod}}(t) = f(t) \cdot \vec{w}, \qquad (17)$$

where $\vec{w}$ is the vector of the weighting factors at the point of contact. $\vec{w}$ is of the same dimensionality as the the vectors of modal displacements $\vec{x}_{\text{mod}}$ and velocities $\dot{\vec{x}}_{\text{mod}}$, i.e. the number of modes. Vice versa, the scalar spatial displacement $x_{\text{con}}$ and velocity $\dot{x}_{\text{con}}$ at the same point are calculated from the modal displacement and velocity vectors by

$$\begin{aligned} x_{\text{con}} &= \vec{w}^T\vec{x}_{\text{mod}}, \\ \dot{x}_{\text{con}} &= \vec{w}^T\dot{\vec{x}}_{\text{mod}}. \end{aligned} \qquad (18)$$

We now return to the example of subsection 2.1 of a point-mass $m_h$ interacting with an object vibrating according to a linear equation (3) with the interaction force modelled according to equation (9). The vibrating object shall however now be given in abstract form as by equations (16), (17) and (18). For a description of the scenario during contact we again combine the displacements of the vibrating object and of the point-mass "hammer" into a new displacement vector $\acute{\vec{x}} := (\vec{x}_{\text{mod}}\ x_{n+1})^{\text{T}}$. In order to formulate the equations in terms of this new vector we now have to insert equation (18) into (10) and replace equation (11) by the vector equation

$$\ddot{\vec{x}}_{\text{mod}}(t) = \vec{f}^{\text{int}}_{\text{mod}}(t) + \vec{f}^{\text{ext}}_{\text{mod}}(t), \qquad (19)$$

with $\vec{f}^{\text{int}}_{\text{mod}}(t)$ denoting the internal forces inside the modal object depending on $D_k$ and $D_c$. For $\vec{f}^{\text{ext}}_{\text{mod}}(t)$ we finally have to insert (9) *with a negative sign* (compare subsection 2.1) into equation (17) and replace $x_{l_{\text{con}}}$ by (18). When tracing these replacements we get an equation as in (12) where $K_{\text{con}}$ is now found to be

$$K_{\text{con}} := k_{\text{con}} \cdot \begin{pmatrix} \vec{w} \\ \frac{-1}{m_h} \end{pmatrix} \cdot \left(\vec{w}^T, -1\right)$$

$$= k_{\text{con}} \cdot \begin{pmatrix} w_1 w_1 & w_1 w_2 & \dots & w_1 w_n & -w_1 \\ w_2 w_1 & w_2 w_2 & \dots & w_2 w_n & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n w_1 & w_n w_2 & \dots & w_n w_n & -w_n \\ -\frac{w_1}{m_h} & \dots & & -\frac{w_1}{m_h} & \frac{1}{m_h} \end{pmatrix} \quad (20)$$

and $C_{\text{con}}$ analogous with $k_{\text{con}}$ replaced by $c_{\text{con}}$. We are now in the same situation as in subsection 2.1 and may apply the procedure described in subsection 2.2.

Figure 2 shows the temporal trajectory (x-axis in samples at rate $44100$ Hz) of a point-mass bouncing under the influence gravity on a surface. The vibratory behaviour of the surface at the point of contact is described by an abstract modal object with modal frequencies of $220$Hz, $950$Hz and $3500$Hz. In figure 3 the behaviour of the overall energy of the system is plotted over time (same time scale as in figure 2 a) ). This overall energy consists of the sum of potential and kinetic energy of the free mass as well as the vibrating object and the energy stored in the "contact spring". At each contact energy is transferred from the free mass to the modal object. The artefacts of energy loss at the end of each contact (compare subsection 2.2) are too small to be resolved in this plot. It can be seen (also at any zooming stage) that the overall system energy decays monotonically. As explained in subsection 2.2 effects of the discrete-time implementation never lead to an accidential increase of energy. The algorithm is thus always stable and effects such as never-ending bouncing behaviour due to computational artefacts can not occur. It can be seen in figure 2c) that finally continuous contact is reached. The algorithm thus allows to model scenarios where both distinct impacts and continuous contact may occur, such as in sliding or rolling interaction.

## 3. CONCLUSIONS

A discrete-time algorithm modelling contact of solid objects has been developed. It is based on a model of both, the involved objects and the interaction force and therefore allows to model dynamic situations of repeated and continuous contact. By applying the modal approach to both phases, of contact and without, system energy is controlled and an accurate, economic and stable discrete-time implementation is reached. The presented general technique may be applied to a wide range of scenarios of contacting solid objects, also such of continuous interaction such as rolling or sliding, which suffer from energetic instabilities of previous contact models.

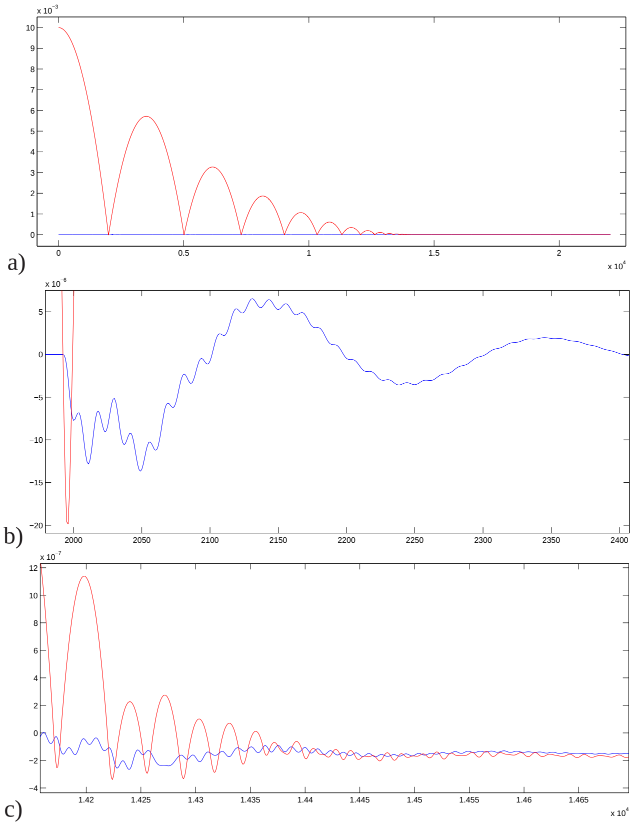


Figure 2: *Temporal trajectory of a point-mass (red) hitting a vibrating object under influence of gravity. Global bouncing behaviour a), detail of one single impact b); c) shows the detail of the last single bounces until continous contact is reached.*

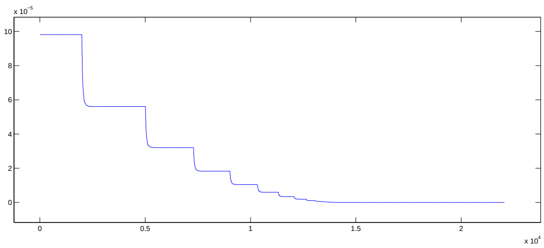

Figure 3: *Temporal decay of the overall energy of the system of a point-mass falling onto a vibrating object.*

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