DOPPLER EFFECT OF A PLANAR VIBRATING PISTON: STRONG SOLUTION, SERIES EXPANSION AND SIMULATION

Tristan Lebrun

S3AM team, STMS, IRCAM-CNRS 9912-UPMC
1 place Igor Stravinsky, 75004 Paris, France
tristan.lebrun@ircam.fr

Thomas Hélie

S3AM team, STMS, IRCAM-CNRS 9912-UPMC
1 place Igor Stravinsky, 75004 Paris, France
thomas.helie@ircam.fr

ABSTRACT

This article addresses the Doppler effect of a planar vibrating piston in a duct, as a plane wave radiation approximation generated by a loudspeaker membrane. This physical model corresponds to a nonlinear problem, because the linear propagation is excited by a moving boundary condition at the piston face: this introduces a varying propagation time between the piston and a fixed receiver. The existence of a regular function that solves the problem (a so-called "strong" solution) is proven, under a well-posed condition that guarantees that no shock occurs. This function satisfies an implicit equation to be solved. An algorithm based on the perturbation method is proposed, from which an exact solution can be built using power series. The convergence of the power series is numerically checked on several examples. Simulations derived from a truncated power series provide sound examples with audible intermodulation and distortion effects for realistic loudspeaker excursion and speed ranges.

1. INTRODUCTION

The Doppler effect in loudspeakers is due to the membrane motion: this introduces a varying propagation time between the piston and a fixed receiver. For large frequency range speakers, this results in distortion effects. This phenomenon has been hightlighted by Beers and Belar [1], who recommend the use of a multi-way system to reduce its influence.
In [2], Butterweck proposes a simplified 1D model based on plane wave propagation generated by a moving piston. He proposes to approximate the solution by a truncated series expansion, from which a criterion to evaluate the distortion is built.

This paper restates the 1D model introduced in [2] and investigates the well-posedness of the problem. A necessary and sufficient condition is presented for the existence of a regular solution, characterized by an implicit equation. This equation admits a unique solution, and its regularity order is proven to be related to that of the membrane displacement function. The proof relies on the method of characteristics. This so-called moving boundary problem has already been investigated, and analytical solutions have been established [3, 4]. In this paper, infinite regular displacement functions are considered, and the perturbation method proposed by [2] is adopted to derive the power series expansion in an exact recursive way. The series expansion corresponds to a Volterra series and its terms involve the partial Bell polynomials. Finally, simulations are carried out by truncation of the series expansion and both harmonic and intermodulation distortions are evaluated.
This paper is organized as follows. Section 2 introduces the physical model and the equations to solve. Then Section 3 presents the strong solution derived from the method of characteristics, and a recursive algorithm based on power series expansion is described in Section 4. Finally the simulations are presented in Section 5.

2. PROBLEM STATEMENT

2.1. Description

Consider a semi-infinite duct excited by a vibrating planar piston (see Figure 1), which follows those four hypotheses:

(H1) conservative linear acoustic plane waves propagation in an adiabatic homogeneous gas initially at rest,

(H2) no wave coming from the right side of the duct,

(H3) no shockwave propagates,

(H4) piston position described by the function $t \mapsto \xi(t)$ at the left side of the duct, initially at rest $\big(\xi(t) = 0$ for $t \leq 0\big)$.

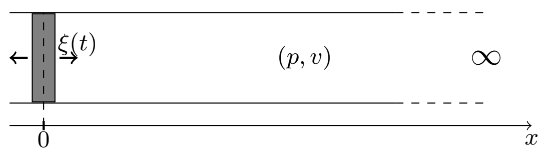


Figure 1: *A piston vibrates around $x = 0$ in a semi-infinite duct.*

2.2. Physical model

Following **(H1)**, the wave propagation in the duct is described by

$$\rho_0 \partial_t v(x,t) + \partial_x p(x,t) = 0, \quad (1)$$

$$\partial_t p(x,t) + \rho_0 c_0^2 \partial_x v(x,t) = 0, \quad (2)$$

where p and v denote the acoustic pressure and the particle velocity, x and t are the space and time variables and ρ_0 and c_0 are the air density and the sound velocity, respectively.
The general solution (1-2) is decomposed into traveling waves as follows

$$\begin{aligned} v(x,t) &= v^+(t - x/c_0) - v^-(t + x/c_0), \\ p(x,t) &= \rho_0 c_0 \big(v^+(t - x/c_0) + v^-(t + x/c_0)\big), \end{aligned}$$

where functions $t \mapsto v^{\pm}(t)$ represent forward and backward waveforms.

Condition **(H2)** implies that there is no backward wave ($v^- = 0$), so that

$$v(x,t) = v^+(t - x/c_0), \tag{3}$$
$$p(x,t) = \rho_0 c_0 v^+(t - x/c_0), \tag{4}$$

only propagates from left to right.

Condition **(H3)** implies that v^+ must be a regular function. In the following, this class of solutions is called the class of "strong solutions" (contrary to the case of weak solutions that can admit jumps or singularities).

Finally, because of **(H4)**, the particle velocity at position ξ is the piston velocity ξ'. This means that the waves are driven by the piston face according to

$$v\big(\xi(t),t\big) = \xi'(t), \tag{5}$$

and that they propagate from left to right in the space-time domain

$$\mathbb{K}_\xi = \{(x,t) \in \mathbb{R}^2 \text{ s.t. } x \geq \xi(t)\}, \tag{6}$$

according to (3-4).

The problem described by **(H1-H4)** can be restated and reduced to the following question: given a piston motion function $t \mapsto \xi(t)$, does there exist a regular waveform function $t \mapsto v^+(t)$ such that

$$v^+\big(t - \xi(t)/c_0\big) = \xi'(t), \tag{7}$$

and such that, on domain (6), (3-4) is a solution of (1-2) ?

3. EXISTENCE OF STRONG SOLUTIONS

This section addresses the existence of regular solutions based on the method of characteristics. It provides a necessary and sufficient condition on $\mathcal{C}^1$-regular functions ξ.
In the following, the exponent "a" denotes quantities related to the arrival time of the acoustic wave at the observer point, whereas "d" denotes the departure time of the wave from the piston face.

Definition 1 (Characteristic) *Let us define the regular functions*

$$\begin{aligned} \tau^a_\xi : \quad & \mathbb{K}_\xi \mapsto \mathbb{R} \\ & (x,t) \mapsto t + \frac{x - \xi(t)}{c_0}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} K_\xi : \quad & \mathbb{K}_\xi \mapsto \mathbb{K}^a_\xi \\ & (x,t) \mapsto \big(x, \tau^a_\xi(x,t)\big) \end{aligned} \tag{9}$$

where the image set $\mathbb{K}^a_\xi$ *is*

$$\mathbb{K}^a_\xi = \Big\{\big(x, \tau^a_\xi(x,t)\big) \ \textit{for}\ (x,t) \in \mathbb{K}_\xi\Big\} \tag{10}$$

In this definition (see Figure 2), $T = \tau^a_\xi\big(x = X, t = \theta\big)$ provides the arrival time $t = T$ at position $x = X$ of an acoustic wave emitted at time $t = \theta$ and position $x = \xi(\theta)$, according to (3). Indeed, the particle velocity v at $(x,t) = (X,T)$ is

$$v^+(\tau^a_\xi(X,\theta) - X/c_0) = v^+(\theta - \xi(\theta)/c_0),$$

which corresponds to the particle velocity at $(x,t) = \big(\xi(\theta),\theta\big)$.

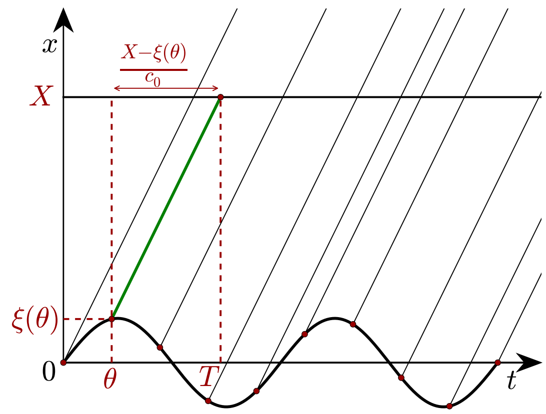


Figure 2: *Illustration of the characteristics for a sinusoidal piston motion: the lines $L_\xi(t) = \{K_\xi(x,t),\ x \geq \xi(t)\}$ are the space-time locus on which the acoustic waves are constant. This figure details the case for the wave emitted at $(x,t) = \big(\xi(\theta),\theta\big)$ and arriving at $(x,t) = K_\xi(X,\theta) = (X,T)$, with $T = \tau^a_\xi(X,\theta)$.*

Figure 3 is a 3D illustration of the wave propagation from the piston face $\xi(t)$ to the observer point x through the characteristic K_ξ. The piston displacement $\xi(t)$ is a sine function and its velocity $\xi'(t)$ describes an helicoidal trajectory, highlighting the moving character of the boundary condition.
This results in various lengths of characteristic lines (represented by arrows) and so various times of propagation to reach the position x. Thus, the waveform of the particle velocity observed at x is not the exact copy of the piston motion.

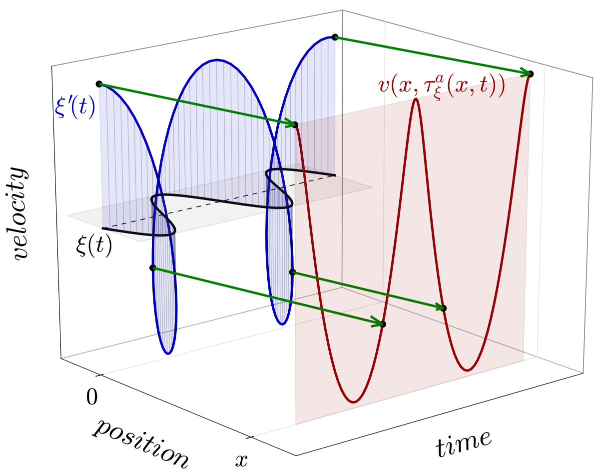


Figure 3: *Perspective view of an acoustic wave generated at $\big(\xi(t),t\big)$ and observed at $\big(x,\tau^a_\xi(x,t)\big)$. The piston displacement is described by the curve labelled $\xi(t)$, and the velocity by the helicoidal trajectory $\xi'(t)$. Four lines of characteristic K_ξ are represented by arrows, connecting $\xi'(t)$ with the particle velocity v at position x.*

The characteristic defined in (9) is based on the emission time of the acoustic wave (propagation occurs at time $\tau_\xi^a(x,t) > t$). Then the expression of the particle velocity observed at x is $v\big(x, \tau_\xi^a(x,t)\big)$. However we seek a solution based on the observation time, so that the velocity is $v(x,t)$. Thus, the existence of strong solutions involves the reciprocal application K_ξ^{-1}.

The main result of this section is given by the following theorem.

Theorem 1 (Strong solution)**.** *Let ξ be a $\mathcal{C}^{n+1}$-regular function with $n \in \mathbb{N}$. Suppose that its (signed) Mach number is stricly less than 1, namely,*

$$\forall t \in \mathbb{R}, \quad \xi'(t)/c_0 < 1. \tag{11}$$

Then, the following results hold:

(i) K_ξ is a $\mathcal{C}^n$-regular diffeormorphism from $\mathbb{K}_\xi$ to $\mathbb{K}_\xi^a$,

(ii) The strong solution v of (3-6-7) is $\mathcal{C}^n$-regular and given by

$$\begin{array}{rcl} v : \mathbb{K}_\xi^a & \longrightarrow & \mathbb{R} \\ (x,t) & \longmapsto & \xi' \circ \tau_\xi^d(x,t) \end{array} \tag{12}$$

where the (unique) departure time $\tau_\xi^d(x,t)$ of the travelling wave observed at (x,t) is given by the $\mathcal{C}^n$-regular function

$$\begin{array}{rcl} \tau_\xi^d : \mathbb{K}_\xi^a & \longrightarrow & \mathbb{R} \\ (x,t) & \longmapsto & [K_\xi^{-1}(x,t)]_2, \end{array} \tag{13}$$

where $[K_\xi^{-1}(x,t)]_2$ denotes the second component of the function.

Consequently, if ξ is $\mathcal{C}^\infty$-regular, then all the above-defined functions are also $\mathcal{C}^\infty$-regular.

The proof of this theorem is given in appendix B. It relies on the two following propositions.

Proposition 1 (K_ξ is a bijection)**.** *Let ξ be a $\mathcal{C}^1$-regular function. Then, function $K_\xi : \mathbb{K}_\xi \to \mathbb{K}_\xi^a$ is a bijection if function ξ satisfies condition (11).*

Proof. (i) By construction of the image set $\mathbb{K}_\xi^a$ (see (10)), K_ξ is a surjective function.
(ii) Injectivity. ξ is a $\mathcal{C}^1$-regular function, so that τ_ξ^a and then K_ξ are also $\mathcal{C}^1$-regular functions. For all $(x,t) \in \mathbb{K}_\xi$, the jacobian of K_ξ is given by

$$J_{K_\xi}(x,t) = \begin{pmatrix} 1 & 0 \\ 1/c_0 & 1 - \xi'(t)/c_0 \end{pmatrix},$$

in which $1 - \xi'(t)/c_0$ is strictly positive. Therefore $K_\xi(x,t)$ is strictly increasing with respect to x and t. The monotonicity proves that K_ξ is injective, which concludes the proof. □

Remark 1 (Condition on the Mach number)**.** *(11) is a sufficient condition to ensure the bijectivity of K_ξ. To obtain a necessary and sufficient condition, (11) must be modified as follows:*

- *the set $E = \{t \in \mathbb{R} \text{ s.t. } \xi'(t)/c_0 = 1\}$ has a zero measure, and*
- *$\xi'(t)/c_0 < 1$ for all $\in \mathbb{R} \setminus E$.*

Indeed, as E has a zero measure, the monotonicity is still fulfilled so that this condition is still sufficient. This condition is also necessary because if E has a non zero measure, there exists an open subset of $\mathbb{K}_\xi$ on which the jacobian of K_ξ has a zero determinant, making K_ξ locally constant and so non injective.

Remark 2 (Relations between $\mathbb{K}_\xi^a$ and $\mathbb{K}_\xi$)**.** *By construction of $\mathbb{K}_\xi^a$ (see (10)), $\mathbb{K}_\xi \subseteq \mathbb{K}_\xi^a$: the characteristics lines cover $\mathbb{K}_\xi$ (see $L_\xi(t)$ in Figure 2.) Moreover, if the Mach number condition (11) is fulfilled, the characteristics lines $L_\xi(t)$ start from the boundary $\partial\mathbb{K}_\xi$ of $\mathbb{K}_\xi$ at point $\big(\xi(t), t\big), t \in \mathbb{R}$ and do not recross this boundary $\partial\mathbb{K}_\xi = \big(\xi(t), t\big), t \in \mathbb{R}$. Therefore, $\mathbb{K}_\xi^a = \mathbb{K}_\xi$.*

Proposition 2 (K_ξ is a $\mathcal{C}^n$-diffeomorphism)**.** *Let ξ be a $\mathcal{C}^1$-regular function that satisfies condition (11). If function ξ is also $\mathcal{C}^{n+1}$-regular with $n \in \mathbb{N}$, then function K_ξ is a $\mathcal{C}^n$-regular diffeomorphism. Consequently, if ξ is $\mathcal{C}^\infty$-regular, then K_ξ is a $\mathcal{C}^\infty$-regular diffeomorphism.*

Proof. Let $n \in \mathbb{N}$ and consider $\xi \in \mathcal{C}^{n+1}$-regular. Suppose that condition (11) is satisfied. Then, from Proposition 1, function $\tau_\xi^d(x,t) : (x,t) \in \mathbb{K}_\xi \mapsto [K^{-1}(x,t)]_2 \in \mathbb{R}$ exists and is continuous.
Now, we prove by induction that τ_ξ^d is $\mathcal{C}^p$-regular for $1 \le p \le n$.

• Case $p = 1$: $\tau_\xi^d \in \mathcal{C}^1$.
From (8), τ_ξ^a is $\mathcal{C}^1$-regular so that $D_2\tau_\xi^a$ is continuous, where D_2 stands for the derivative with respect to the second component of the function. Moreover, $\forall(x,t) \in \mathbb{K}_\xi^a$, $\big(x, \tau_\xi^d(x,t)\big) \in \mathbb{K}_\xi$ and $f(x,t) = D_2\tau_\xi^a\big(x, \tau_\xi^d(x,t)\big) = 1 - \xi'\big(\tau_\xi^d(x,t)\big)/c_0$ defines a function $f : \mathbb{K}_\xi^a \longrightarrow \mathbb{R}$ which is:

(i) continuous, because it is a composition of the continuous functions τ_ξ^d and $D_2\tau_\xi^a$,

(ii) strictly positive because (11) is satisfied.

The jacobian of K_ξ^{-1} given by

$$\begin{array}{rl} J_{K^{-1}} : & \mathbb{K}_\xi^a \longrightarrow \mathcal{M}_{2,2}(\mathbb{R}) \\ & (x,t) \longmapsto \begin{pmatrix} 1 & 0 \\ \frac{-1}{c_0 f(x,t)} & \frac{1}{f(x,t)} \end{pmatrix}, \end{array} \tag{14}$$

is then a continuous function. Hence, K_ξ^{-1} and then τ_ξ^d (see (13)) are $\mathcal{C}^1$-regular.

• Case $p \ge 2$: If τ_ξ^d is $\mathcal{C}^p$-regular, then τ_ξ^d is $\mathcal{C}^{p+1}$-regular.
From (14), the jacobian $J_{K^{-1}}$ is $\mathcal{C}^p$-regular. Therefore τ_ξ^d is $\mathcal{C}^{p+1}$-regular, which concludes the proof. □

From Proposition 1, there exists a function $\tau_\xi^d : \mathbb{K}_\xi^a \mapsto \mathbb{R}$, such that for all $(x,t) \in \mathbb{K}_\xi^a$, equation (8) reads

$$\tau_\xi^d(x,t) = t - \frac{x + \xi \circ \tau_\xi^d(x,t)}{c_0}. \tag{15}$$

The calculation of this space-time function can be reduced to that of a simpler time function, because of the following property.

Property 1 (Translational symmetry)**.** *Let $\delta > 0$ and $\mathcal{T}_\delta : (x,t) \mapsto (x + \delta, t + \delta/c_0)$ be the space-time translation operator of space-time shift $(\delta, \delta/c_0)$.*

Then, for any arrival space-time point $(x,t) \in \mathbb{K}^a_\xi$, the translated point $\mathcal{T}_\delta(x,t)$ is in $\mathbb{K}^a_\xi$ and the departure time of this translated point is unchanged:

$$\forall (x,t) \in \mathbb{K}^a_\xi,\ \delta > 0, \quad \tau^d_\xi \circ \mathcal{T}_\delta(x,t) = \tau^d_\xi(x,t). \tag{16}$$

and $v\big(\mathcal{T}_\delta(x,t)\big) = \xi' \circ \tau^d_\xi(x,t)$.

The proof is straightforward from (3).
Consequently, the next section addresses the derivation of a solver for the reduced problem given by

$$\tau_\xi(t) = t + \frac{\xi \circ \tau_\xi(t)}{c_0}, \tag{17}$$

where $\tau_\xi(t) = \tau^d_\xi(x=0,t)$, and from which the strong solution is derived

$$v(x,t) = \xi'\big(\tau_\xi(t) - x/c_0\big). \tag{18}$$

Remark 3 (Equivalent Eulerian description)**.** *Consider function $V : (x,t) \mapsto \xi'\big(\tau_\xi(t - x/c_0)\big)$ defined on $\mathbb{R}^2$. This function coincides with v on $\mathbb{K}^a_\xi$ (see (12)) and is such that $V \circ \mathcal{T}_\delta$ is invariant with respect to δ. Consequently, function V extends solution v on the space-time domain $\mathbb{R}^2$ and $t \mapsto V(0,t)$ stands for the equivalent source description at $x = 0$ (Eulerian description).*

4. RECURSIVE METHOD BASED ON POWER SERIES

First, a dimensionless version of the model is established, so that the representation as power series is simplified. Second, a method based on series expansion is described, in order to carry out computation of (17-18).

4.1. Dimensionless model

Consider the following change of variable, given in Table 1.

Table 1: *Dimensionless Model.*

Variables	Functions
$\tilde{x} = x/c_0$	$\tilde{p} = p/(\rho_0 c_0)$
$\tilde{t} = t$	$\tilde{v} = v/c_0$
	$\tilde{\xi} = \xi/c_0$

Replacing equations (3-6-7) by their versions denoted with a "tilde" yields

$$\tilde{p}(\tilde{x},\tilde{t}) = \tilde{v}(\tilde{x},\tilde{t}) = \tilde{v}^+(\tilde{t} - \tilde{x}), \tag{19}$$

the time-space propagation domain becomes

$$\tilde{\mathbb{K}}_\xi = \{(\tilde{x},\tilde{t}) \in \mathbb{R}^2 \text{ s.t. } \tilde{x} > \tilde{\xi}(\tilde{t})\}, \tag{20}$$

and the left boundary

$$\tilde{v}^+\big(\tilde{t} - \tilde{\xi}(\tilde{t})\big) = \tilde{\xi}'(\tilde{t}). \tag{21}$$

The strong solution becomes

$$\tilde{v}(\tilde{x},\tilde{t}) = \tilde{\xi}'\big(\tilde{\tau}_\xi(\tilde{t}) - \tilde{x}\big) \in (\tilde{\mathbb{K}}_\xi, \mathbb{R}), \tag{22}$$

where

$$\tilde{\tau}_\xi(\tilde{t}) = \tilde{t} + \tilde{\xi} \circ \tilde{\tau}_\xi(\tilde{t}). \tag{23}$$

For the sake of readability, the symbols "tilde" are omitted in the sequel.

4.2. Perturbation method and series expansion

Consider $\xi \in \mathcal{C}^\infty$ (so that $\tau_\xi \in \mathcal{C}^\infty$ from the theorem 1), and let

$$\epsilon(t) = \tau_\xi(t) - t, \tag{24}$$

which corresponds to the variation of propagation time due to source motion. Reformulating (22-23) with $\epsilon(t)$ yields

$$v(x,t) = \xi'\big(t - x + \epsilon(t)\big), \tag{25}$$

$$\epsilon(t) = \xi\big(t + \epsilon(t)\big). \tag{26}$$

Now, we solve the implicit equation (26) using a perturbation method. Consider that (26) describes a system of input ξ and output $\epsilon(t)$, and apply the change of variable $\xi = \alpha.u$, with $\alpha \geq 0$. The method consists in writing the output as a formal power series in α, such that

$$\epsilon(t) = \sum_{n=0}^{\infty} \frac{\epsilon_n(t)}{n!}\alpha^n. \tag{27}$$

Then (26) becomes

$$\sum_{n=0}^{\infty} \alpha^n \frac{\epsilon_n(t)}{n!} = \alpha.u\Big(t + \sum_{n=0}^{\infty} \alpha^n \frac{\epsilon_n(t)}{n!}\Big). \tag{28}$$

Now function u is developped into Taylor series at point t:

$$\sum_{n=0}^{\infty} \alpha^n \frac{\epsilon_n(t)}{n!} = \alpha \sum_{n=0}^{\infty} \frac{u^{(n)}(t)}{n!}\Big(\sum_{m=0}^{\infty} \alpha^m \frac{\epsilon_m(t)}{m!}\Big)^n. \tag{29}$$

Given that the right-hand side of (29) is multiplied by α, the term $\epsilon_0(t)$ (corresponding to α^0) is vanished. Then (29) reads

$$\sum_{n=1}^{\infty} \alpha^n \frac{\epsilon_n(t)}{n!} = \alpha \sum_{n=0}^{\infty} \frac{u^{(n)}(t)}{n!}\Big(\sum_{m=1}^{\infty} \alpha^m \frac{\epsilon_m(t)}{m!}\Big)^n. \tag{30}$$

The right-hand side is a series composition and can be developped using the Faá di Bruno power series formula:

$$\sum_{n=1}^{\infty} \frac{\epsilon_n(t)}{n!}\alpha^n = \alpha.u(t) + \sum_{n=1}^{\infty} c_n(t).\alpha^{n+1}, \tag{31}$$

where

$$c_n(t) = \sum_{k=1}^{n} \frac{u^{(k)}(t)}{n!} B_{n,k}\big(\epsilon_1(t), \epsilon_2(t), ..., \epsilon_{n-k+1}(t)\big) \tag{32}$$

and $B_{n,k}$ are the partial Bell polynomials.

Now,

$$\sum_{n=1}^{\infty} \alpha^n \big[\frac{\epsilon_n(t)}{n!} - u(t)\delta_{1,n} - c_{n-1}(t)\big] = 0, \tag{33}$$

hence, for all $\alpha > 0$,

$$\begin{aligned} &\epsilon_1(t) = \xi(t), \\ &\epsilon_n(t) = n.\sum_{k=1}^{n-1} \xi^{(k)}(t) B_{n-1,k}\big(\epsilon_1(t), ..., \epsilon_{n-k}(t)\big), \\ &\quad \forall n > 1, \end{aligned} \tag{34}$$

where the amplitude α has been dropped, so that $u = \xi$.

Property 2 (Multivariate polynomial)**.** *$\epsilon_n(t)$ has the form*

$$\mathcal{C}_n: \quad \epsilon_n(t) = P_n\big(\xi(t), \xi'(t), ..., \xi^{(n-1)}(t)\big), \tag{35}$$

where P_n is a homogeneous multivariate polynomial of degree n.

The proof of this property is given in Appendix C.

Now, the expression of the particle velocity (25) is also expanded into Taylor series, at point $t - x$:

$$v(x,t) = \sum_{n=0}^{\infty} \frac{\xi^{(n+1)}(t-x)}{n!} \Big(\sum_{m=1}^{\infty} \frac{\epsilon_m(t)}{m!} \Big)^n. \tag{36}$$

Applying again the Faá di Bruno formula for power series composition finally yields the expression of the particle velocity,

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t), \tag{37}$$

where

$$\begin{aligned} v_1 &= \xi^{(1)}(t-x), \\ v_n &= \sum_{k=1}^{n-1} \frac{\xi^{(k+1)}(t-x)}{k!(n-1)!} B_{n-1,k}\big(\epsilon_1, ..., \epsilon_{n-k}\big), \\ &\forall n > 1. \end{aligned} \tag{38}$$

The first orders of v_n are listed below, where the dimensionless model has been dropped:

$$v_1(x,t) = \xi^{(1)}(t - x/c_0), \tag{39a}$$

$$v_2(x,t) = \frac{1}{c_0}.\xi(t - x/c_0).\xi^{(2)}(t - x/c_0), \tag{39b}$$

$$\begin{aligned} v_3(x,t) = \frac{1}{c_0^2}\Big[&\frac{1}{2}\xi(t - x/c_0)^2.\xi^{(3)}(t - x/c_0) \\ &+ \xi(t - x/c_0).\xi^{(1)}(t - x/c_0).\xi^{(2)}(t - x/c_0)\Big]. \end{aligned} \tag{39c}$$

Remark 4 (Link with Volterra series)**.** *From the property 2, $\epsilon_n(t)$ has homogeneous order n with respect to $\xi(t)$ and its derivatives. Then the system of input $\xi(t)$, on which the perturbation method is applied, and output $\epsilon(x,t)$ (and by extension $v(x,t)$) can be represented by a Volterra series expansion [5], formally in the space of distributions.*

Remark 5 (Convergence)**.** *Some results about the convergence of Volterra series are available in [6, 7] for L^∞ input signals: they involve L^1 Volterra kernels. In the present problem, the convergence and the class of admissible inputs are a more complicated issue: because of the time-derivative in (38), kernels are not in L^1. In addition to (11), some properties about asymptotic behaviour (bounds on time derivatives or on frequency characteristics) have to be examined to set the convergence condition: this future work will provide the class of admissible waveforms.*

Remark 6 (Practical implementation)**.** *For implementation purpose, the expansion (37) is truncated at a given order N. Moreover, for a signal in the frequency range $f_{\max}$, a frequency oversampling by a factor of N is applied so that $f_s = N.(2f_{\max})$, ensuring no aliasing (the frequency range of v_n is then $nf_{\max} \le Nf_{\max}$).*

4.3. Numerical evaluation of the convergence

Although the convergence domain of (38) is not tackled from the theoretical side, this section presents simulations of the acoustic output for a 40Hz, 1s sinusoidal velocity input at various Mach numbers. Figure 4 shows the results for different truncation orders, from $N = 1$ to $N = 15$.
Divergence of the series is noted at Mach 1 and above, which is consistent with the condition of existence of the strong solution (11). Since the usual range of membrane velocities is far below this limit, the convergence criterion should be met, at least for sinusoidal motions.

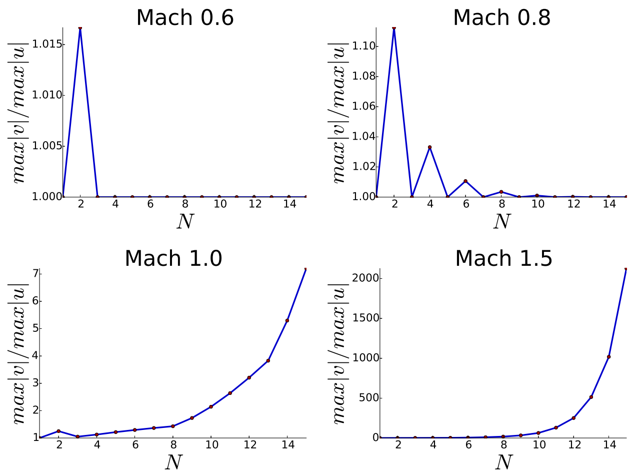


Figure 4: *Convergence computation for various sinusoidal input velocities. Simulations at Mach 0.6 and Mach 0.8 converge, contrary to simulations at Mach 1 and Mach 1.5.*

5. SIMULATION AND DISTORTION EVALUATION

5.1. Harmonic distortion

The harmonic distortion is evaluated for a sinusoidal piston velocity with constant amplitude over the frequency spectrum (loudspeaker with a flat frequency response), so that the input displacement takes the form

$$\xi(t) = \frac{A}{2\pi f_1} \sin(2\pi f_1 t). \tag{40}$$

Two simulations are implemented, listed in Table 2 below. The velocity amplitude A corresponds to a high velocity of the driver's membrane, T is the simulation duration and N is the truncation order of (38). Results are presented in Figures 5and 6.

Table 2: *Simulation parameters used for harmonic distortion simulations.*

Name	f_1	A	T	N	fs
HD1	40 Hz	1 m/s	50 s	3	$N \times 44100$ Hz
HD2	1 kHz	1 m/s	50 s	3	$N \times 44100$ Hz

• The same harmonic distortion is observed for **HD1** and **HD2**, respectively in Figure 5 and Figure 6, with the apparition of the second harmonic around -55dB. Injecting the expression of $\xi(t)$ in the first orders equations (39) leads to

$$\begin{aligned} v_1(x,t) &= A\sin(2\pi f_1 t), \\ v_2(x,t) &= \frac{A^2}{2c_0}\big(1-\cos(4\pi f_1 t)\big). \end{aligned}$$

It appears that harmonic distortion only depends on piston velocity amplitude, as confirmed by the simulations. Moreover, the amplitude ratio between v_1 and v_2 is of -56,7 dB, which is consistent with the numerical result.

• A THD[1] of 0.14 % is noted. In most cases, harmonic distortion can then be neglected. This is in agreement with previous studies [2, 8]. A truncation at order 2 appears to be sufficient to characterise this type of distortion (in the loudpseaker velocities range).

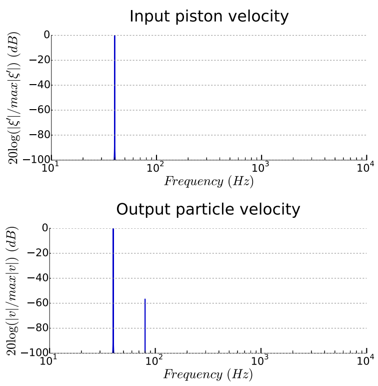


Figure 5: *Results of simulation* ***HD1****. The amplitude spectrum of the input piston velocity (top) and the ouput particle velocity (bottom) are shown. Harmonic distortion is observed at* $2f_1$.

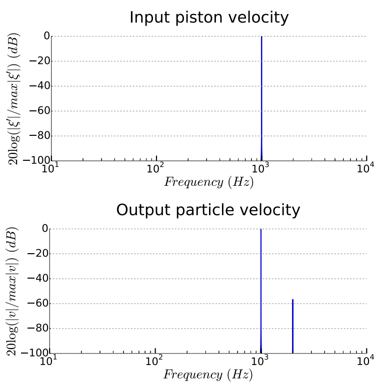


Figure 6: *Results of simulation* ***HD2****. The amplitude spectrum of the input piston velocity (top) and the ouput particle velocity (bottom) are shown. Harmonic distortion is observed at* $2f_1$.

[1]Total Harmonic Distortion is calculated as the ratio of RMS amplitude of the harmonics to the RMS amplitude of the fundamental.

5.2. Intermodulation distortion

In this section, the intermodulation distortion is examined by simulating the particle velocity for an input displacement of the form

$$\xi(t) = \frac{A}{2\pi f_1}\sin(2\pi f_1 t) + \frac{A}{2\pi f_2}\sin(2\pi f_2 t).$$

Given the weak amplitude of harmonic distortion, it is assumed that both phenomena can be analyzed separately (harmonic distortion that occurs in the following computations is neglected). The simulation parameters are listed in Table 3. Simulation results are presented in Figures 7 and 8.

Table 3: *Simulation parameters used for intermodulation distortion simulation.*

Name	f_1	f_2	A	T	N	fs
IMD1	40 Hz	3 kHz	1 m/s	50 s	5	$N \times 44100$ Hz
IMD2	40 Hz	6 kHz	1 m/s	50 s	5	$N \times 44100$ Hz

• An increase of intermodulation distortion is observed between both simulations with a noticeable gain in sidebands amplitude for **IMD2**. The Intermodulation Factors[2] are 7.5% and 11.4% for **IMD1** and **IMD2**, respectively. This confirms that the intermodulation effect, which rises with the value of f_2, should be taken into account for large frequency range speakers.

• Truncation at order 5 was sufficient for both simulations to capture the distortion phenomena in a dynamic of 100dB. However, higher orders might be necessary in case of sound synthesis with audio signal as input, since intermodulation distortion can be much higher with complex signals.

• Finally, Figure 9 shows the intermodulation distortion for simuation **IMD2** in the time domain, resulting in phase modulation of the high frequency component.

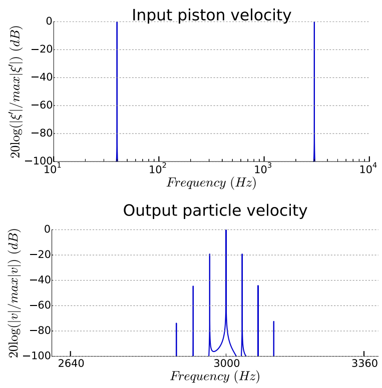


Figure 7: *Amplitude spectrum of the particle velocity (bottom) for simulation* ***IMD1*** *(top). The bottom figure is rescaled around the frequency of interest,* f_2*. Intermodulation distortion is observed at* $f_2 - p.f_1$ *and* $f_2 + p.f_1$ *for* $p = 1, 2, 3$.

[2]Intermodulation Factor is calculated as the ratio of the RMS amplitude of the sidebands to the RMS amplitude of the carrier frequency.

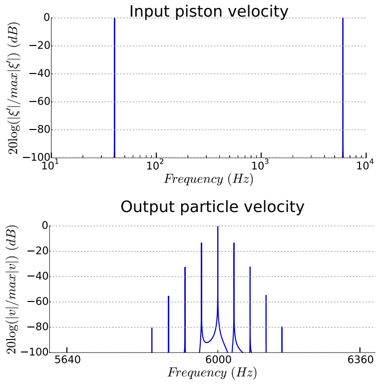
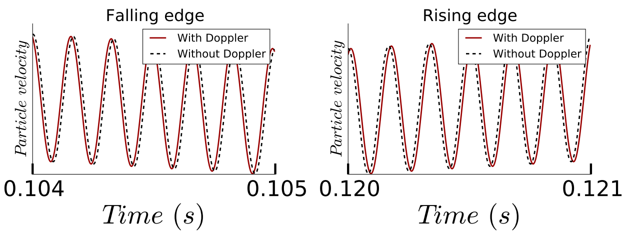


Figure 8: *Amplitude spectrum of the particle velocity (bottom) for simulation **IMD2** (top). The bottom figure is rescaled around the frequency of interest, f_2. Intermodulation distortion is observed at $f_2 - p.f_1$ and $f_2 + p.f_1$ for $p = 1, 2, 3, 4$.*



(a) (b)

Figure 9: *Time samples of simulation **IMD2**. Phase modulation is observed, since the high frequency component is phase-shifted on the falling edges and conversely on the rising edges.*

In order to highlight the influence of Doppler effect on more complex signals, a final simulation is performed with an input signal composed of two chirps (50Hz to 3kHz) and one pure tone (1kHz), at constant amplitude 1 m/s. Spectrograms of the particle velocities are shown without Doppler effect on the upper part of Figure 10, which is an exact copy of the input (delayed with x/c_0). Bottom Figure 10 shows the acoustic output with the Doppler effect, truncated at $N = 5$. Both harmonic and intermodulation distortion are clearly visible.

6. CONCLUSION

The model and simulations presented in this paper confirm that the Doppler effect causes audible intermodulation distortion, as stated in [1, 2, 9]. The numerical examination of expansion terms shows that: (i) the convergence is statisfied quickly, (ii) the first expansion terms can have an audible impact up to the orders 3 or 4.
Future work is concerned with:

- the convergence proof of the series expansion and the establishment of a truncation error bound,
- a corrector that compensates the Doppler effect based on series inversion and the control of the truncation order on the inverse series,
- the improvement of the model by including the convection phenomenon.

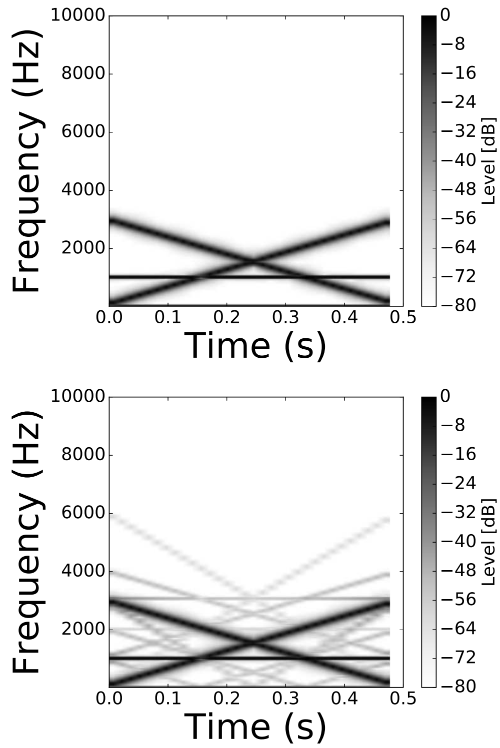


Figure 10: *Spectrograms of the particle velocities, normalized in dB, simulated with (bottom) and without (top) Doppler effect. The input piston velocity is composed of two chirps and one pure tone.*

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B. PROOF OF THE THEOREM 1

Proof. Let ξ be a $\mathcal{C}^{n+1}$-regular function with $n \in \mathbb{N}$ that satisfies (11). Point (i) results from propositions 1 and 2.

Proof of point (ii) is divided into two steps:

a. Boundary condition: $v\big(\xi(t), t\big) = \xi'(t)$

Because of (i), τ_ξ^d and $v = \xi' \circ \tau_\xi^d$ are $\mathcal{C}^n$-regular. From (9), the following equality can be computed:

$$K_\xi\big(\xi(t), t\big) = \big(\xi(t), t\big), \quad \forall t \in \mathbb{R}. \tag{41}$$

Moreover, by definition of the reciprocal function K_ξ^{-1},

$$\forall (x,t) \in \mathbb{K}_\xi^a, \quad (x,t) = K_\xi^{-1} \circ K_\xi(x,t), \tag{42}$$

so that, replacing (x,t) by $\big(\xi(t), t\big)$,

$$\big(\xi(t), t\big) = K_\xi^{-1}\big(\xi(t), t\big). \tag{43}$$

Taking the second component of (43), where $[K_\xi^{-1}]_2 = \tau_\xi^d$, yields

$$\tau_\xi^d\big(\xi(t), t\big) = t. \tag{44}$$

Therefore the solution $\xi' \circ \tau_\xi^d\big(\xi(t), t\big)$ verifies the boundary condition (5).

b. Propagation: $v(x,t) = v^+(t - x/c_0)$

Define the function

$$\begin{aligned} L: \quad & \mathbb{K}_\xi^0 \longmapsto \mathbb{K}_\xi^a \\ & (x,t) \longmapsto (x, t + x/c_0), \end{aligned} \tag{45}$$

where $K_\xi^0 = \big\{(x,t) \in \mathbb{R}^2 | (x, t + x/c_0) \in \mathbb{K}_\xi^a\big\}$.

Moreover, isolating the second component of

$$\begin{aligned} L(x,t) &= K_\xi \circ K_\xi^{-1} \circ L(x,t) \\ &= \Big(x, \tau_\xi^a\big(x, \tau_\xi^d \circ L(x,t)\big)\Big), \end{aligned} \tag{46}$$

leads to, for all $(x,t) \in \mathbb{K}_\xi^0$,

$$t + \frac{x}{c_0} = \tau_\xi^d \circ L(x,t) + \frac{x - \xi \circ \tau_\xi^d \circ L(x,t)}{c_0}. \tag{47}$$

This equation is equivalent to, for all $(x,t) \in \mathbb{K}_\xi^0$,

$$t = F_\xi\Big(\tau_\xi^d \circ L(x,t)\Big), \tag{48}$$

where

$$\begin{aligned} F_\xi: \quad & \mathbb{R} \longmapsto \mathbb{R} \\ & t \longmapsto t - \xi(t)/c_0. \end{aligned} \tag{49}$$

Now, for all $t \in \mathbb{R}$, $F_\xi'(t) = 1 - \xi'(t)/c_0 > 0$ and $\lim_{t \to \pm\infty} F_\xi(t) = \pm\infty$ so that F_ξ is a strictly increasing bijective function. It follows that

$$\tau_\xi^d \circ L(x,t) = F_\xi^{-1}(t), \tag{50}$$

proving that $\tau_\xi^d \circ L(x,t)$ does not depend on x.

Consequently, the strong solution (12) can be written, for all $(x,t) \in \mathbb{K}_\xi^a$,

$$\begin{aligned} v(x,t) &= \xi' \circ \tau_\xi^d(x,t) \\ &= \xi' \circ \tau_\xi^d \circ L(x, t - x/c_0), \end{aligned} \tag{51}$$

which has the form $v^+(t - x/c_0)$, with $v^+ = \xi' \circ \tau_\xi^d \circ L$, that concludes the proof.

□

C. PROOF OF THE PROPERTY 2

Proof. Let us prove by induction the following property

$$\mathcal{C}_n: \quad \epsilon_n = P_n\big(\xi(t), \xi'(t), .., \xi^{(n-1)}(t)\big),$$

where P_n is a homogeneous multivariate polynomial of degree n.

• *Step 1: C_2 is true*, since $\epsilon_2 = \xi(t).\xi'(t)$, which is a polynomial or degree 2.

• *Step 2: If $\mathcal{C}_n$ is true, then $\mathcal{C}_{n+1}$ is true.*

Let $\epsilon_n = P_n(\xi, \xi', ..., \xi^{(n-1)})$. Then

$$\epsilon_{n+1} = (n+1).\sum_{k=1}^{n} \xi^{(k)}.B_{n,k}\Big(\xi(t), .., P_{n-k+1}(\xi, \xi', ..., \xi^{(n-k)})\Big),$$

where the Bell polynomials are developped

$$\begin{aligned} & B_{n,k}\Big(\xi, .., P_{n-k+1}(\xi, \xi', ..., \xi^{(n-k)})\Big) \\ &= \sum_{j_1, \ldots, j_{n-k} \in \Pi_{n-1,k}} \big(\frac{\xi(t)}{1!}\big)^{j_1} \times \ldots \\ & \times \big(\frac{P_{n-k+1}(\xi, ..., \xi^{(n-k)})}{(n-k+1)!}\big)^{j_{n-k+1}}. \end{aligned}$$

and $\Pi_{n,k} = \begin{cases} j_1 + j_2 + \ldots + j_{n-k+1} = k \\ j_1 + 2.j_2 + \ldots + (n-k+1).j_{n-k+1} = n. \end{cases}$

$B_{n,k}$ is a sum over products of polynomials, then it is a polynomial. Moreover the degree of each term of the sum over $\Pi_{n,k}$ is

$$1.j_1 + 2.j_2 + \ldots + j_{n-k+1} + (n-k+1) = n,$$

from the definition of $\Pi_{n,k}$. Therefore the degree of all nonzero terms of ϵ_{n+1} is $n+1$, that concludes the proof. □