# A QUATERNION-PHASE OSCILLATOR

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## ABSTRACT

An approach to designing dynamical systems with a three-dimensional state space is described that can be used to build a variety of non-periodic oscillators. The state space is taken to be a 3-sphere, which is identified with the manifold of unit quaternions. Any such system can be described as a quaternion-valued ordinary differential equation, which is digitally realized using an approximation as a finite difference e quation. Two examples are shown. Compared to previous applications of dynamical systems used to generate audio samples, the approach described here offers a wide choice of specific flows which can neither diverge nor approach a stable limit point.

## 1. INTRODUCTION

An oscillator may be thought of as an unstable feedback system, and, conversely, any unstable feedback system that has a bounded state space can be viewed as an oscillator. This equivalence has long been a rich source of inspiration for electronic musicians; classic examples include *Rain Forest* (David Tudor) [1, 2], *Hornpipe* (Gordon Mumma) [3], *Look at the back of my head for awhile* (Salvatore Martirano) [4], *Pendulum Music* (Steve Reich) [5], and *I am Sitting in a Room* (Alvin Lucier) [6], among many others.

As a way of relinquishing detailed control over musical processes, this tendency or current in electronic music composition can be seen as a reflection of a broader shift in 20th-century "art music" practice. So, for instance, John Cage made extensive use of chance processes as a way to avoid having to assert his own volition during the composition process, much in the same way as the early atonal composers used tone rows and series. (Although the word "chance" in common usage implies randomness, many chance processes did not include any specifically random elements; for instance, Cage's *Freeman Etudes* [7] relies on star charts and Zorn's *Cobra* [8] on unpredictable human group dynamics. Here we will be concerned only with deterministic processes.)

By the early 1970s, when modular analog synthesizer design had reached its mature stage, synthesizer builders had adopted these ideas in their module designs. A well-known example is the Buchla 259 dual analog oscillator [9] in which the two oscillators could be coupled in a variety of ways to yield complex, non-periodic behaviors. Buchla's own performance practice showed transparently his interest in complex, evolving dynamical systems as a catalyst for musical creativity.

In this paper we present a class of algorithms, realized digitally, in which non-periodic oscillators are considered as dynamical systems. The use of dynamical systems that can exhibit periodic, bifurcating-period, and/or chaotic behaviors as generators of musical waveforms is already well known; a representative example is the Chua oscillator [10]. These systems are defined as first-order ODE systems that are thought of as creating a flow within a state space, where the "state" is simply the tuple of variables that are related by the ODE system. For example, the double-well oscillator [11] responds to a forcing input that can determine the frequency of the output, or, if greatly reduced in amplitude, allow the oscillator to settle into one of two possible limit cycles with a different frequency. There is a regime of chaotic behavior between the two extremes.

A quite different, and fruitful, use of dynamical systems as signal generators is the use of ODEs to physically model a vibrating body such as a musical instrument. Here again there is a vast literature. These systems tend to be high-dimensional; for instance the widespread use of delay lines to "model" an air column or taut string can be regarded as an Euler-method solver for an ODE whose state consists of hundreds or thousands of individual points along the air column or string. The literature on this topic is too vast to even summarize here.

There is also a large literature on a related but different topic, that of iterated functions such as circle maps [12]. We will not consider this class of chaotic systems here but note in passing that the ODE approach is inherently continuous in time and can be up- and down-sampled to change pitch and/or limit foldover effects. There is no corresponding way to continuously speed up or slow down the output of an iterated map. The dynamical systems considered here have the general advantage that they are inherently continuous in time and thus are more intuitively controllable than iterated functions, although still not at all straightforward to deal with.

Our approach will be most closely related to that of coupled oscillator systems, which we consider in the following section. A key advantage to this type of system is that it is relatively easy to prevent the system from simply stopping by reaching a stable point. The parameter space of such a system describes how the frequencies of the oscillators should depend on each others' phases, and for this reason it is easier—although not exactly easy—to understand how changing parameters will change the output sound, as compared to more abstractly defined systems such as the Chua oscillator or the Lorenz attractor.

One type of behavior that one could hope for in a musical sound generated by a dynamical system could be changes over longer time frames that somehow relate to short-time behavior [13]. This is motivated by the idea that musical form should reflect musical material, as argued classically by composers such as Varese and Stockhausen. Simply listening to the output of, say, a Lorenz attractor gives the impression of timbral roughness but not of sound that varies over a musical time scale. As we will see, the quaternion-phase oscillator shown here will get us a bit closer to this ideal, and can be made to do so within a relatively low-dimensional state space and parameter space.

## 2. COUPLED OSCILLATORS AS DYNAMICAL SYSTEMS

If we think of a collection of coupled oscillators as a dynamical system, the state space of the dynamical system consists of all possible combinations of phases of the oscillators. So, for example, two coupled digital oscillators can be seen as a discrete-time approximation of a) flow through a state space that can be identified with a torus $\mathbb{S}^1 \times \mathbb{S}^1$ [14], as shown in figure 1. Here the vectors show the velocity of a continuous flow through the state space. In the horizontal region where the phase of the independent oscillator, $x_2$, is near zero, the phase of the dependent one, $x_1$, changes its frequency so that it, likewise, is attracted to the point of zero phase. The extent of the sync band (the region near $x_2 = 0$) and the strength of attraction within it are parameters of the flow, along with the base frequencies of the two oscillators.

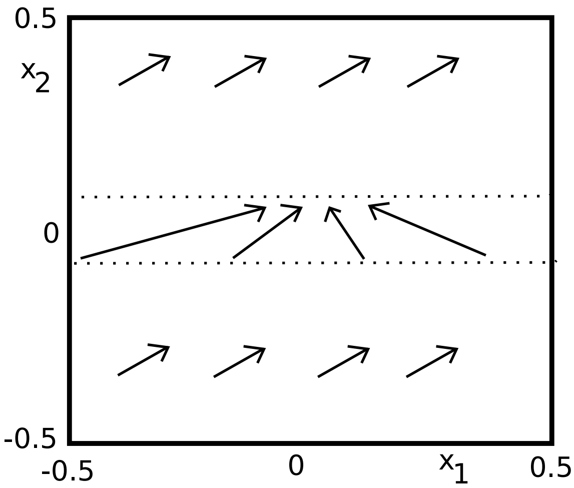


Figure 1: *Flow diagram for a soft-synced oscillator pair. The oscillator with phase $x1$ (horizontal axis) is soft-synced to the other one. The state space is a torus with top/bottom and left/right edges identified.*

This oscillator pair can be theoretically described as a dynamical system:

$$\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right) = v(x_1, x_2) \tag{1}$$

where $v$ is a vector-valued function of the two state variables $x_1$ and $x_2$, and is depicted as the vector field shown in the figure.

If the coupling is sufficiently strong, the dependent oscillator can be phase-locked into a multiple of the frequency of the independent one, so that the output is periodic. This is equivalent to saying that the time-varying state—the ordered pair $(x_1, x_2)$—reaches a limit cycle. If the parameters are chosen to give a weaker coupling between the two oscillators the system can become non-periodic. However, since the state space is two-dimensional and since two different trajectories through state space cannot cross each other, we don't see anything that could be called chaotic behavior. The minimum dimensionality that would permit such behaviors is three.

## 3. THE 3-SPHERE AS STATE SPACE

If we wish to design a dynamical system with a three-dimensional state space, an obvious choice of space would be three-dimensional Euclidean space, $\mathbb{R}^3$. This has the advantage of having a six-dimensional symmetry group (spatial rotations and translations), bringing many mathematical conveniences. But since $\mathbb{R}^3$ is unbounded, it is often difficult to prevent state trajectories from racing off to infinity. On the other hand, no bounded subset of $\mathbb{R}^3$ sports a symmetry group of more than three dimensions, as for example the unit ball does.

Here we will propose the 3-sphere, $\mathbb{S}^3$, as an ideal state space for building chaotic flows. It has the advantages of low dimensionality, compactness, and a large, six-dimensional symmetry group. Furthermore, no topologically based differences between dynamical behaviors in $\mathbb{R}^3$ versus $\mathbb{S}^3$ can arise, since, even though the two spaces are not homeomorphic, we can remove any single point from $\mathbb{S}^3$ that happened not to be hit by a certain trajectory–such a point always exists—to make a space homeomorphic to $\mathbb{R}^3$ after all. This was not the case for our earlier toroidal state space which allows dynamical behaviors that are not possible on, say, a disc. Choosing $\mathbb{S}^3$ as a state space gives us the smallest possible interesting dimensionality and topological complexity and, at the same time, maximum symmetry.

Supposing that we have settled on $\mathbb{S}^3$ as a state space for a dynamical system, one fruitful way to design one and/or study its behavior is to identify the state space with the set of unit quaternions. The set $\mathbb{Q}$ of quaternions is defined as:

$$\mathbb{Q} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\} \tag{2}$$

where $i, j, k$ are all square roots of $-1$, any two of which anti-commute, and

$$ij = k, \quad jk = i, \quad ki = j \tag{3}$$

The norm of a quaternion $q$ is defined as

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2} \tag{4}$$

The subset of $\mathbb{Q}$ where $j = k = 0$ can be identified with the complex plane $\mathbb{C}$. We can describe a continuous-time, complex-valued sinusoidal oscillator as a dynamical system whose state space, $S^1$, is identified with the set of unit complex numbers, whose time evolution obeys the differential equation:

$$\frac{dz}{dt} = f(z) \cdot z \tag{5}$$

where $f(z)$ takes pure imaginary values. Under this condition we can check that

$$\frac{d}{dt}(|z|^2) = 0 \tag{6}$$

so that if the initial state lies on the unit circle, the time-varying state never goes outside it. For the system to oscillate sinusoidally we take $f(z) = i\omega$, independent of $z$, in which case $\omega$ is the angular frequency.

The same thing can be done using quaternions in general. The state of the oscillator is then a time-dependent unit quaternion $q(t)$ (i.e., satisfying $|q(t)| = 1$ for all $t$), that satisfies a differential equation

$$\frac{dq}{dt} = \Omega(q) \cdot q \tag{7}$$

where we now choose $\Omega(q)$ always to take the form

$$\Omega = \omega_i i + \omega_j j + \omega_k k \tag{8}$$

i.e., to have a real part equal to zero. We can identify the triple $(\omega_i, \omega_j, \omega_k)$ as a three-dimensional frequency. In the same way as before we find that

$$\frac{d}{dt}(|q|^2) = 0 \tag{9}$$

so that if $q(0)$ is taken to be a unit quaternion, then the entire trajectory over time will lie on the unit 3-sphere $a^2+b^2+c^2+d^2 = 1$, which is our desired state space.

## 4. SYMMETRIES OF THE STATE SPACE

In order to study the behavior of a quaternion oscillator with fixed quaternion frequency $\omega = (0, \omega_i, \omega_j, \omega_k)$ we will need to invoke the symmetry of the state space, which we now make explicit. For any orthogonal 3-by-3 matrix $\mathcal{O}$, there is an automorphism:

$$A : \mathbb{Q} \to \mathbb{Q} \tag{10}$$

$$\begin{pmatrix} a \\ (b, c, d)^\top \end{pmatrix} \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{O} \end{pmatrix} \begin{pmatrix} a \\ (b, c, d)^\top \end{pmatrix} \tag{11}$$

that leaves the real part $a$ untouched but applies a rotation matrix to the triple $b, c, d$. This is a four-dimensional rotation that leaves the point $(1, 0, 0, 0)$—the quaternion equal to the real number 1—fixed.

In addition there are isometries that preserve addition but not multiplication, which are applied by multiplying by a fixed unit quaternion:

$$T : \mathbb{Q} \to \mathbb{Q} \tag{12}$$

$$q \hookrightarrow q_0 q \tag{13}$$

where $|q_0|^2 = 1$. These are also rotations, and any rotational symmetry of the unit sphere $|q| = 1$ can be decomposed as the composition $TA$ of an automorphism followed by multiplication by a constant.

## 5. FIXED FREQUENCY QUATERNION OSCILLATOR

Returning to the continuous-time oscillator with fixed quaternion frequency $\Omega = (0, \omega_i, \omega_k, \omega_k)$, we choose an automorphism $A$ that maps $\Omega$ to $(0, \omega, 0, 0)$ where

$$\omega = |\Omega| = \sqrt{\omega_i^2 + \omega_j^2 + \omega_k^2} \tag{14}$$

The resulting differential equation is reduced to the earlier complex-valued one whose solution is

$$q(t) = \exp(i\omega t)q(0) = (\cos\omega + i\sin\omega)q(0) \tag{15}$$

Multiplying by any unit complex number acts on a quaternion as a rotation in the $(1, i)$ plane and an equal rotation in the $(j, k)$ plane. Applying the inverse automorphism shows that oscillation at a generally chosen quaternion frequency can be decomposed as simultaneous sinusoidal oscillation in two oblique planes that are perpendicular to each other, at the frequency given by equation 14 .

The intrepid reader will now see the possibility of frequency modulation–in which the three-component quaternion frequency varies in time—changing not only the resultant frequency $\omega$ but also the orientation of the planes of rotation. Phase modulation can be generalized to multiply the oscillator's quaternion-valued output on the left and/or right by time-varying unit quaternions, without affecting the oscillator's internal state. We will leave these avenues unexplored for now.

## 6. DISCRETE TIME

In practice we don't wish to use the closed-form solution of equation 15, since it only holds if the frequency is constant in time, and neither do we wish to try to solve the differential equation numerically: we want a discrete-time recurrence relation. Once again we work by analogy with the simpler complex-valued oscillator to propose the recurrence relation:

$$q[n] = r(q[n-1]) \cdot q[n-1] \tag{16}$$

where $r(q)$ is a unit quaternion that depends on the state $q$. Comparing this with the continuous-time solution of equation 15, we get that

$$r(q) \approx \exp(\tau\Omega) \tag{17}$$

where $\tau$ is the sample period.

Although this invocation of the exponential function is perfectly legitimate (it can be defined via the differential equation itself, or as a Taylor series, or as a limiting product of numbers near 1), here we simply evaluate it by applying the reverse of the automorphism that took us into the complex plane. A solution to the complex recurrence is given by

$$z[n] = \cos(\omega\tau) + i\sin(\omega\tau) \tag{18}$$

and since the inverse automorphism sends

$$(0, 1, 0, 0) \hookrightarrow (0, \omega_i, \omega_j, \omega_k)/\omega \tag{19}$$

we get

$$r(q) \approx \cos(\omega\tau) + \frac{\sin(\omega\tau)}{\omega} \cdot (0, \omega_i, \omega_j, \omega_k) \tag{20}$$

To realize the oscillator we simply plug this formula for $r$ into the recurrence relation (equation 16). The oscillator's raw output is in the form of four audio signals that may be played in different loudspeakers, or, additionally or alternatively, may be subjected to any desired waveshaping function. In practice it is usually enough to send two of the four state variable to left and right channels of a stereo pair; the resulting signal gets louder and softer as the total instantaneous signal power $|q|^2 = 1$ is variously distributed among the four components.

## 7. SOME PRACTICAL CONSIDERATIONS

The phase spaces of a dynamical system can often offer one or more points of stability, toward which a path can be fatally attracted. The 3-sphere has the convenient property that it admits continuous flow velocities that never vanish. (This is not possible, for example, on the 2-sphere). If the flow velocity, or equivalently the quaternion frequency, is a continuous function of the state, it suffices that the frequency be nonzero everywhere for there to be no points of stabililty.

We will also be interested in frequencies that are discontinuous functions of the state space, in which additional conditions are necessary to prevent the state getting caught in a discontinuity; for instance, two or more regions could meet at a point in state space toward which paths converge from all directions. We can prevent this from happening by constraining the three frequency components to be nonnegative everywhere, and to have a norm that is always positive but also never exceeding $\pi$ radians per sample. Quaternion frequencies $\Omega$ greater than $\pi$ in length are analogous to frequencies greater than the Nyquist in an ordinary, complex-valued signal.

Although there is no obvious correlate to the Nyquist theorem for quaternion-valued signals—the proof of the Nyquist theorem blithely assumes that multiplication is commutative—we can at least observe that for a constant quaternion frequency $\Omega$, each of the four components of the quaternion phase is a sinusoid whose frequency is $|\Omega|$, and so in this simple case we must keep the norm of the quaternion frequency below $\pi$.

Although I don't know of any proof, I can offer a further conjecture about quaternion frequency functions obeying the above constraints (never zero; individual components always nonnegative; norm below $\pi$): that such functions allow no path to lie entirely within any one open hemisphere. This conjecture motivated the examples developed below.

That the frequency has three distinct components raises a question: is a time-varying quaternion frequency $\Omega(t)$ directly observable? The answer is yes and no. Any attempt to generalize Fourier analysis to allow multiple-component frequencies would require a multidimensional time axis, which we don't have in practice. On the other hand, if we are presented with the phase of a particular quaternion oscillator, we can simply divide two successive samples to find an instantaneous quaternion frequency. In principle at least, in this situation one could hear the distinction between different quaternion frequencies.

Due to numerical accuracy limitations, in practice the norm of the computed state of a quaternion oscillator will slowly drift. (This was already an issue for a complex-valued oscillator computed using the recurrence relation). To prevent this we renormalize the state at each time step.

Alternatively, the oscillator can be converted into a nonlinear filter by applying a gain smaller than one to the quaternion phase at each step of the recursion, and adding a term for a filter input. If we insert a saturation function the gain may be set greater than one, and the system becomes an oscillating filter analogous to the Moog ladder filter [15, 16].

## 8. EVALUATION BY EXAMPLE

The reader or listener should be warned that the author has a weakness for ornery, glitchy sounds. The techniques shown here are quite capable of yielding more mellifluous results than are shown here, but a thorough exploration of those possibilities is left for another time and/or another investigator.

We start by considering how to exploit the symmetries of the phase space to obtain a seven-dimensional parameter space that can give rise to usefully variable sounds. We observe that the path generated by a fixed quaternion frequency is a circle. Except in special cases, exactly half of this circle lies inside any fixed hemisphere. If we now fix two different quaternion frequencies to hold inside and outside the hemisphere, the path will describe two (usually different) semicircles, both meeting at the two points of intersection with the boundary, normally at two different speeds.

So far we have only made a periodic waveform which could be made by many other means. To make the example interesting, we take two different hemispheres, which, with their complements, cut the phase space into four regions. Again hewing to the simplest reasonable scenario, we let the quaternion frequency alternate between two fixed ones, $\Omega_1$ and $\Omega_2$.

We now exploit the symmetries of the phase space to place the two hemispheres in a specific position: first, multiply each point by a fixed quaternion to make the first hemisphere align with the $i, j, k$ subspace, by moving one of its poles to the quaternion 1. A phase $q$ lies in this hemisphere or its complement depending on the signum of $\mathrm{Re}(q)$.

The other hemisphere pair is then described by the signum of a multiple of $q$, say $\eta q$. We now exploit the $(i, j, k)$ automorphism to choose $\eta$ to lie in the $(1, i)$ plane, so that the only free parameter is its angle of inclination. The original path can thus be transformed into one that is described by two quaternion frequencies and one angle of inclination.

We further simplify our example by setting the angle of inclination at $\pi/2$ radians (i.e., $\eta = i$) and so dividing the space into four equal quadrants. Moreover, since we still have not used the $(j, k)$ component of the automorphism, we could now invoke that to place an additional constraint on $\Omega_1$ and $\Omega_2$ to reduce the number of parameters to five, but there is no clear principle guiding the choice so for now we leave the number of parameters at six.

One complication creeps into the realization of this oscillator: the frequency switches discontinuously as the phase crosses hemisphere boundaries, and the exact time of the transition may strobe audibly against the DSP sample rate. This can be quite audible if one or both quaternion frequencies is high in magnitude. To ameliorate this situation we design a soft crossover function whose maximum slope is controlled by one additional parameter, which we call the crossover slope. The resulting parameter space is seven-dimensional, consisting of two quaternion frequencies and the crossover slope.

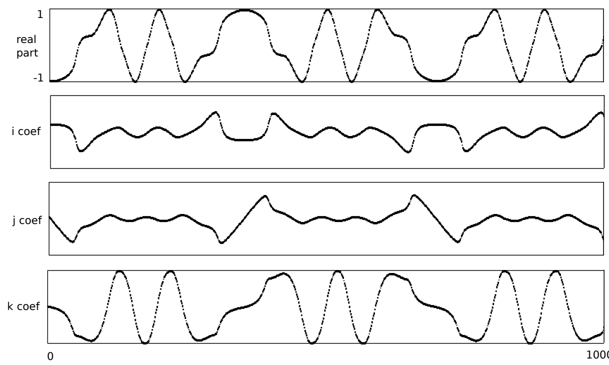


Figure 2: *The evolution of the four components of a quaternion oscillator, plotted in time. Vertical ranges are from -1 to 1; the (horizontal) time axis is in samples at an arbitrary rate.*

Figure 2 shows the four components of the oscillator phase, as a function of sample number, in our example with one possible choice of parameters. The time units are quite arbitrary. In practice one can change the speed globally by multiplying the quaternion frequency by a constant factor. In the realization described here, the six frequency parameters are controlled as percentages of a base frequency which is specified separately, which for convenience is given in continuously variable "floating-point MIDI" units.

Figure 3 shows the same example as a path through phase space. The coordinate axes are assigned to the $j$ and $k$ components. The 1 component (the real part) is indicated by the size

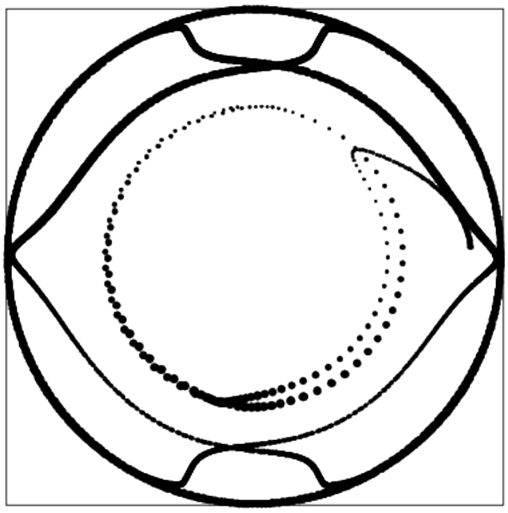

Figure 3: *The same as figure 2, but with the $j$ and $k$ components graphed on the horizontal and vertical axes. The size of the dots indicate value of the real part (the "1" term), ranging from -1 (smallest) to 1 (largest).*

of the drawn points (which during occasional fast transitions are separately visible but run together much of the time and show up as as a thickening of the path instead). Unsurprisingly, the two-dimensional projection shows a sequence of ellipsoidal arcs, since the quaternion frequency changes by steps as a hemisphere is entered or exited.

This signal was generated using a Pure Data demonstration patch uploaded to the directory
`msp.ucsd.edu/ideas/quaternion-osc/`.
A subdirectory contains three stereo soundfiles that show off other types of behavior that would be less amenable to graphing. The parameters are, first, a reference frequency given in midi units; next, the two quaternion frequencies (six components) given as percentages of the reference frequency; and finally the crossover slope, usually between 1 and 100.

This example shows some behaviors similar to that of other known chaotic dynamical systems. For example, the Lorenz attractor can be used to generate sound directly by taking everyone's favorite parameters $\rho = 28, \sigma = 10, \beta = 8/3$, and choosing a time scale so that the x, y, and z components of the state are heard audibly. The result is a dirty but partly pitched sound. This is audibly somewhat like the "nasty clarinet sound" settings on the sample patch.

The present network can exhibit interesting limit cycle behaviors. As figure 3 suggests, limit cycles can give rise to complicated-looking periodic waveforms, which may have symmetries coming from those of the underlying space and choice of quaternion frequencies—for instance, the graphed waveform has only odd harmonics. It is also fairly easy to find settings with at least two different limit cycles with different waveforms and frequencies, such as the third parameter set in the demonstration patch. This behavior resembles that of the double-well oscillator.

The last sound example shows that behavior approaching limit cycles can be entertainingly complex. This is obtained in the demonstration patch by switching from either of the first four parameter sets to the fifth one.

It should be acknowledged that, even in this very simple case, it is exceedingly hard to understand how to change the parameters to obtain some specifically desirable result. So far, the only proven method for searching for parameters is trial and error.

## 9. MUSICAL APPLICATION

One desirable quality of an unstable system would be that it exhibit audibly unpredictable results over a range of different time scales. This is not a property either of the demonstration patch, or of the two prior examples considered. One reason for this is that, with only four regions cut off by hemispheres, any trajectory is likely to frequently hit all four regions. If we wish for the path to sometimes visit certain regions and sometimes different ones, we could simply increase the number of regions and give each one its own quaternion frequency. This was done as part of an original music production which we will describe here.

The resulting piece, *Your microphone appears to be noisy*, by the *Higgs whatever* (Kerry Hagan and Miller Puckette), is available on
`msp.ucsd.edu/media/music/2020.10.29.higgs-whatever-noisy-mic.mp4`.
It is an eight-minute improvised duo in which each player controls a separate quaternion oscillator. The controls are mapped from acoustic sources, a guitar and a clarinet, whose sounds are only momentarily heard directly.

As in the simpler example described above, the quaternion frequency was held constant except upon crossing hemispherical boundaries. To the "1" and "i" boundaries we added "j" and "k" ones. Rather than specify a separate quaternion frequency for all 16 regions they cut out, we specify a base frequency as a triple, and four additional frequencies to be added on the positive 1, $i$, $j$, and $k$ hemispheres, each constrained to be in a particular direction so that only one parameter is needed for each.

Leaving out the crossover slope for simplicity and letting $s$ denote the unit step function

$$s(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

the quaternion frequency is set to

$$\Omega(q) = \Omega_0 + (0, p_1 s(q_1), p_i s(q_i), p_j s(q_j) + p_k s(q_k)) \quad (22)$$

where $(q_1, q_i, q_j, q_k)$ denote the components of the time-varying state $q$ and $p_1$, ..., $p_k$, are nonnegative parameters controlling the magnitude of additional quaternion frequencies to add when in the $(q_1 \geq 0)$, ..., $(q_k \geq 0)$ hemispheres.

That entering and exiting the 1-hemisphere affects the $i$ component of the quaternion frequency, and so on, is an arbitrary choice. In general, for each hemisphere we could add a three-parameter, general quaternion frequency. Furthermore, the choice of four hemispheres as regions in which to add frequency is arbitrary. They are chosen to be hemispheres to maximize the likelihood that any given path will frequently cross boundaries, but it might be even more interesting to allow some of the regions to be smaller in volume and to affect the path more sparsely in time.

Figures 4 and 5 show one possible behavior of the resulting system, which is also included as a second demonstration patch on

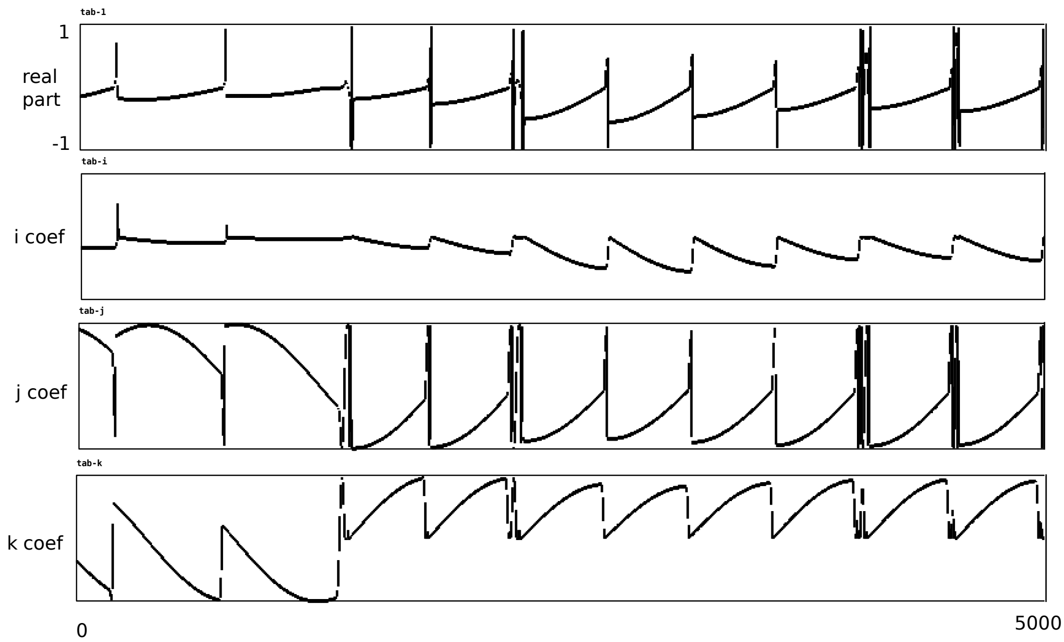


Figure 4: *time-varying quaternion phase for a real example; same units as before.*

the website. Figure 4 in particular suggests that the time patterns are now highly irregular over both shorter and somewhat longer time scales, as we had wished.

It is possible also with this patch to further increase the time instability of the outputs by choosing such a high crossover slope that sample rate strobing becomes noticeable, particularly since the chaotic nature of the flow allows small variations in periodicity to have major effects on subsequent time evolution. This was unwittingly a factor in the piece. The presence of strobing can be checked simply by lowering the base frequency. If the behavior is the same (only slowed down) then strobing is not a major contributor to the sound quality, and otherwise it is. One can always hear the non-strobed "truth" if one slows the system down sufficiently.

## 10. CONCLUSION

The main desiderata expressed in this paper are to achieve a wide range of musical timbres using a coherent and low-dimensional dynamical system; to allow for intuitive exploration of the parameter space; and to give rise to both timbral variety (considered as short-time behavior) and behaviors over musically perceptible time spans, which might thus be somehow perceptually linked. None of these goals can be assessed using a quantitative test. We can at best offer subjective assessments of the relative advantages and disadvantages of one dynamical system over another one.

As compared to other known dynamical systems that generate audio signals, we are able to exhibit many of the same fundamental behaviors such as chaotic motion and limit cycle bifurcation. Somewhat more tenuously, we can claim to easily find parameters leading to "interesting" behaviors over longer time spans. This behavior is also possible to get with classical coupled-oscillator systems.

The quaternion oscillator combines the advantages of a low-dimensional and topologically simple state space, with the availability of a large class of possible flows on that space—any integrable function whose three components are nonnegative and never all zero, and whose norm is below the Nyquist frequency—any one of which will at least oscillate and never limit to a stable point. Within this huge range of possibility we easily found one low-dimensional parameter space that gives rise to sounds interesting enough to use in a finished piece of music.

## 11. REFERENCES

[1] Nicolas Collins, "Composers inside electronics: Music after David Tudor," *Leonardo Music Journal*, vol. 14, no. 1, pp. iv–1, 2004.

[2] John Driscoll and Matt Rogalsky, "David Tudor's *Rainforest*: an evolving exploration of resonance," *Leonardo Music Journal*, vol. 14, pp. 25–30, 2004.

[3] Gordon Mumma, *Cybersonic Arts: Adventures in American New Music*, University of Illinois Press, 2015.

[4] David Rosenboom, "Exploring compositional choice in the SalMar construction and related early works by Salvatore Martirano," *Between the Tracks: Musicians on Selected Electronic Music*, pp. 113–151, 2020.

[5] Steve Reich, "*Pendulum Music* (1968)," *Writings on Music*, vol. 2000, pp. 31–32, 1965.

[6] Alvin Lucier, *I am sitting in a room*, vol. 1013, Lovely Music, 1970.

[7] James Pritchett, "The completion of John Cage's *Freeman Etudes*," *Perspectives of new music*, pp. 264–270, 1994.

[8] John Brackett, "Some notes on John Zorn's *Cobra*," *American Music*, vol. 28, no. 1, pp. 44–75, 2010.

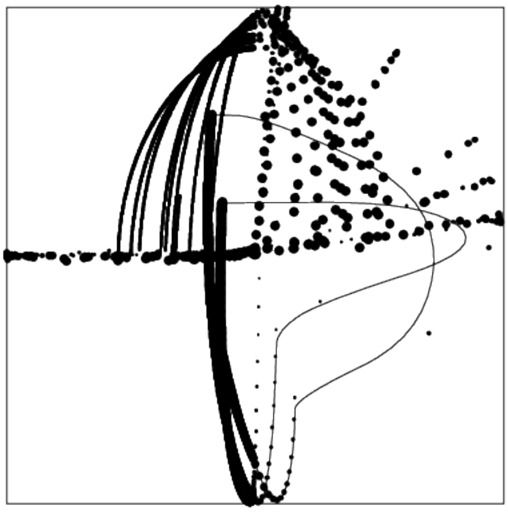

Figure 5: *The same as figure 4, but with the $j$ and $k$ components graphed on the horizontal and vertical axes. The size of the dots indicate value of the real part (the "1" term), ranging from -1 (smallest) to 1 (largest).*


[9] Fabián Esqueda, Henri Pöntynen, Vesa Välimäki, Julian D Parker, et al., "Virtual analog Buchla 259 wavefolder," in *Proceedings of the International Conference on Digital Audio Effects (DAFx), Edinburgh, UK*, 2017, pp. 5–9.

[10] Xavier Rodet, "Nonlinear oscillations in sustained musical instruments: Models and control," *Proc. Euromech, Hamburg, Germany*, 1993.

[11] Nick Collins, "Errant sound synthesis," in *ICMC*, 2008.

[12] Georg Essl, "Circle maps as a simple oscillators for complex behavior: II. experiments," in *Proc. of the 9th Int. Conf. on Digital Audio Effects (DAfX-06)*. Citeseer, 2006, pp. 193–198.

[13] Greg Surges, Tamara Smyth, and Miller Puckette, "Generative audio systems using power-preserving all-pass filters," *Computer Music Journal*, vol. 40, no. 1, pp. 54–69, 2016.

[14] David Medine, "Dynamical systems for audio synthesis: Embracing nonlinearities and delay-free loops," *Applied Sciences*, vol. 6, no. 5, pp. 134, 2016.

[15] Robert A Moog, "A voltage-controlled low-pass high-pass filter for audio signal processing," in *Audio Engineering Society Convention 17*. Audio Engineering Society, 1965.

[16] Timothy E Stinchcombe, "Analysis of the Moog transistor ladder and derivative filters," Tech. Rep., Citeseer, 2008.